岐阜市立女子短期大学研究紀要第 52 輯（2003 年 3 月）

Evaluation of the Trace Anomaly in the U(1) Gauge Field Theory by Mathematica

Koichi SEO

Abstract

A computational method by Mathematica developed by the author is extended to evaluate the trace anomaly for the U(1) gauge field in the gravitational background field. In the general covariant gauge, the differential operator in the exponent of the heat kernel becomes non-diagonal with respect to the Lorentz indices, and this makes the calculation complicated. The result agrees with Endo's one, and is dependent on the gauge parameter. The gauge dependence must be cancelled by adding a local counter term. Besides the DeWitt-Schwinger coefficients after taking the trace with respect to the Lorentz indices, we first present the gauge dependent DeWitt-Schwinger coefficients before taking the trace.

I Introduction

In the covariant gauge, the differential operator in the heat kernel becomes non-diagonal with respect to the Lorentz indices. In literature[1)] , this kind of differential operator is called non-minimal. To avoid this complication, the trace anomaly in the gauge field theory has been evaluated[2)] in the Feynman gauge where the differential operator reduces to the minimal one.

Endo[3)] has derived the exression for the trace anomaly in the general covariant gauge in four space-time dimensions at first with a clever technique. He discovered the gauge parameter dependence of the trace anomaly, and he found this gauge dependence can be eliminated by adding a local counter term. Gusyuin and Gorbar[4)] performed the direct computation of the DeWitt-Schwinger coefficients a_1 in the general covariant gauge and reproduced Endo's result in four space-time dimensions. To our knowledge, a_2, which is related to the trace anomaly in four space-time dimensions, has not been evaluated by any direct computation.

In the previous papers[5)] we proposed a method to evaluate the trace anomaly of a scalar or a Dirac field theory in the gravitational background field with the help of Mathematica[6)]. The heat kernel coefficients were evaluated in a straightforward method. M.J. Booth[7)] has developed a program to compute the heat kernel coefficients in scalar field theories based on Avramidi's algorithm[8)]. Our method was more straightforward and simpler than his method. In the present paper, we extend our method to a U(1) gauge field theory in the general covariant gauge. We confirm Endo's result in four space-time dimensions by a direct computation. Furthermore we present the expression of the DeWitt-Schwinger coefficients before taking the trace with respect to the Lorentz indices at first.

The determinant to be considered is the following one,

$$e^{iW[\,g^{\mu\nu}]} \equiv \int [\,dA_\mu\ d\bar{c}\ dc\,]\, e^{i\int d^n x\, \mathcal{L}}\ , \tag{1.1}$$

with

$$\mathcal{L} \;=\; \sqrt{-g}\,\Big\{-\frac{1}{4}(F_{\mu\nu})^2 - \frac{1}{2\alpha}\,(\nabla_\mu A^\nu)^2 + i\bar{c}\,\Box\, c\Big\}\ , \tag{1.2}$$

where α is the gauge parameter. The Faddeev-Popov term must be taken into account even the ghost fields are decoupled from the gauge fields, since the ghost fields are coupled to the background gravitational fields.

The classical action is not invariant under the local Weyl transformation even in four space-time dimensions, due to the gauge fixing term and the ghost term. The variation of the effective action under the local Weyl transformation have two origins: One is due to the non-invariance of the classical action, and the other comes from the introduction of a cuttoff parameter ϵ to regularize the effective action.

We regularize the effective action by cutting off the integration region around zero of the heat kernel as follows:

$$W[\,g^{\mu\nu}] \;=\; -\frac{i}{2}\,\mathrm{Tr}\Big[\int_\epsilon^\infty \frac{dt}{t}\, e^{-tY^{(A)}}\Big] +i\,\mathrm{Tr}\Big[\int_\epsilon^\infty \frac{dt}{t}\, e^{-tY^{(c)}}\Big]\,, \tag{1.3}$$

where

$$(Y^{(A)})_{\mu\nu} \;=\; g_{\mu\nu}\Box - \nabla_\nu\nabla_\mu + \frac{1}{\alpha}\nabla_\mu\nabla_\nu \tag{1.4}$$

$$Y^{(c)} \;=\; \Box\,. \tag{1.5}$$

The variation of the effective action (1.3) under an infinitesimal local Weyl transformation $g_{\mu\nu} \rightarrow g_{\mu\nu}e^{-2\alpha(x)}$ is related to the vacuum expectation value of the trace of the energy-momentum tensor as

$$\begin{aligned} & W[\,(1+2\alpha)g^{\mu\nu}] - W[\,g^{\mu\nu}] \\ = & \int d^n x\, 2\,\alpha(x)\, g^{\mu\nu} \frac{\delta W}{\delta g^{\mu\nu}} \\ = & \int d^n x \sqrt{-g}\,\alpha(x)\, g^{\mu\nu} < T_{\mu\nu}(x) > , \end{aligned} \tag{1.6}$$

where the energy-momentum tensor is defined by

$$T_{\mu\nu} = T^{(A)}_{\mu\nu} + T^{(c)}_{\mu\nu}\,, \tag{1.7}$$

with

$$\begin{aligned} T^{(A)}_{\mu\nu} &= -g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} + \frac{1}{4} g_{\mu\nu} (F_{\rho\sigma})^2 \\ &\quad + \frac{1}{\alpha}\{A_\mu \nabla_\nu \nabla_\rho A^\rho + A_\nu \nabla_\mu \nabla_\rho A^\rho\} \\ &\quad - \frac{1}{\alpha}\, g_{\mu\nu}\{A^\rho \nabla_\rho \nabla_\sigma A^\sigma + \frac{1}{2}(\nabla_\rho A^\rho)^2\}, \end{aligned} \tag{1.8}$$

$$T^{(c)}_{\mu\nu} = i\left(-\partial_\mu \bar{c}\, \partial_\nu c - \partial_\nu \bar{c}\, \partial_\mu c + g_{\mu\nu}\, \partial^\rho \bar{c}\, \partial_\rho c\right). \tag{1.9}$$

On the other hand, the variation of W is evaluated as

$$\begin{aligned} & W[\,(1+2\alpha)g^{\mu\nu}] - W[\,g^{\mu\nu}] \\ = & \; i\, \mathrm{Tr}\,[\,\alpha e^{-\epsilon Y^{(A)}}] - 2i\, \mathrm{Tr}\,[\,\alpha e^{-\epsilon Y^{(c)}}] \\ & + (\text{canonical trace terms}) \\ = & \; i \int d^n x\, \alpha(x) \lim_{x' \to x} \Big\{ \mathrm{tr}\big[e^{-\epsilon Y^{(A)}} I(x,x') \delta^{(n)}(x,x')\big] \\ & \qquad -2\, e^{-\epsilon Y^{(c)}} \delta^{(n)}(x,x') \Big\} \\ & + (\text{canonical trace terms})\,, \end{aligned} \tag{1.10}$$

where tr stands for the trace with respect to the Lorentz indices, and $I(x,x')$ is the geodesic parallel displacement bivector and its definition and the coincidence limits of its derivatives are listed in Appendix A. Canonical trace terms are originated from the non-invariance of the classical action, and their derivation is given in Appendix B.

Comparing Eq.(1.6) with Eq.(1.10), we obtain the formal expression of the trace anomaly as follows:

$$\begin{aligned} & \sqrt{-g}\, g^{\mu\nu} < T_{\mu\nu}(x) >_{anomaly} \\ = & \; i \lim_{x' \to x} \Big\{ \mathrm{tr}\big[e^{-\epsilon Y^{(A)}} I(x,x') \delta^{(n)}(x,x')\big] \\ & \qquad -2\, e^{-\epsilon Y^{(c)}} \delta^{(n)}(x,x') \Big\}. \end{aligned} \tag{1.11}$$

As explained in the previous paper, we use the following covariant expression for the delta function,

$$\delta^{(n)}(x,x') = \int \frac{d^n k}{(2\pi)^n} e^{i k_a \sigma^a(x,x')}, \tag{1.12}$$

The expression for the trace anomaly (1.11) is also derivable from the path integral measure[9].

II Perturbative expansion

Like the scalar theory or the spinor theory, we first commute the factor $e^{ik\cdot\sigma}$ with Y,

$$\begin{aligned} & (Y^{(A)})_{\mu\nu}\, e^{i k_a \sigma^a} \\ = & \; e^{i k_a \sigma^a} \{ g_{\mu\nu} g^{\rho\sigma} (-i\kappa_\rho + \nabla_\rho)(-i\kappa_\sigma + \nabla_\sigma) \\ & -(-i\kappa_\nu + \nabla_\nu)(-i\kappa_\mu + \nabla_\mu) \\ & + \frac{1}{\alpha}(-i\kappa_\mu + \nabla_\mu)(-i\kappa_\nu + \nabla_\nu) \} \\ = & \; e^{i k_a \sigma^a} \big(K - iX^{(A)} + Y^{(A)}\big)_{\mu\nu}\,, \end{aligned} \tag{2.1}$$

with

$$\kappa_\mu = -k_a \sigma_\mu{}^a\,, \tag{2.2}$$

$$(K)_{\mu\nu} = -g_{\mu\nu}\kappa^2 + \gamma\, \kappa_\mu \kappa_\nu \tag{2.3}$$

$$\gamma \equiv 1 - \frac{1}{\alpha} \tag{2.4}$$

$$\begin{aligned} (X^{(A)})_{\mu\nu} &= g_{\mu\nu}(\kappa^\rho \nabla_\rho + \nabla_\rho \kappa^\rho) \\ &\quad -\gamma(\kappa_\mu \nabla_\nu + \nabla_\mu \kappa_\nu)\,. \end{aligned} \tag{2.5}$$

Exponentiating the relation (2.1), we have

$$e^{-tY^{(A)}} e^{i k_a \sigma^a} = e^{i k_a \sigma^a} e^{-t(K - iX^{(A)} + Y^{(A)})}\,. \tag{2.6}$$

By regarding K as the free Hamiltonian, we apply the perturbation theory,

$$\begin{aligned} & e^{-t(K+V)} = e^{-tK} \\ & + \iint_0^t dt_1\, dt_2\, \delta(t - t_1 - t_2)\, e^{-t_1 K}\, V\, e^{-t_2 K} \\ & + \iiint_0^t dt_1\, dt_2\, dt_3\, \delta(t - \sum_{i=1}^{3} t_i) e^{-t_1 K} V e^{-t_2 K} V e^{-t_3 K} \\ & + \cdots\,. \end{aligned} \tag{2.7}$$

In the scalar theory or the spinor theory, we have moved the factor $e^{t\kappa^2}$ to the leftmost position at first. This time we move the covariant derivative ∇_μ to the rightmost position at first. Then we can take the coincidence limit of e^{-tK}, and it reduces to

$$\begin{aligned} & e^{t(g_{\mu\nu}k^2 - \gamma k_\mu k_\nu)} \\ = & \; \big(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\big)\, e^{tk^2} + \frac{k_\mu k_\nu}{k^2}\, e^{\frac{t}{\alpha}k^2}\,. \end{aligned} \tag{2.8}$$

In order to restrict the tensor structures to be appeared during the calculation, we make use of the following formula,

$$\begin{aligned} & \int \cdots \int_0^t \cdots dt_i \cdots \delta(t - \cdots - t_i - \cdots) \\ & \times \cdots [\,\nabla_\mu,\, e^{-t_i K}\,] \cdots \\ = & -\int \cdots \int_0^t \cdots dt_i \cdots \delta(t - \cdots - t_i - \cdots) \iint_0^{t_i} dt'_i\, dt''_i \\ & \times \delta(t_i - t'_i - t''_i) \cdots e^{-t'_i K}\, [\,\nabla_\mu,\, K\,]\, e^{-t''_i K} \cdots \\ = & -\int \cdots \int_0^t \cdots dt'_i\, dt''_i \cdots \delta(t - \cdots - t'_i - t''_i - \cdots) \\ & \times \cdots e^{-t'_i K}\, [\,\nabla_\mu,\, K\,]\, e^{-t''_i K} \cdots, \end{aligned} \tag{2.9}$$

with

$$\begin{aligned} & -[\nabla_\mu, (K)_{\nu\rho}] \\ &= 2g_{\nu\rho}\,\kappa_\sigma\nabla_\mu\kappa^\sigma - \gamma\,(\nabla_\mu\kappa_\nu\,\kappa_\rho + \kappa_\nu\nabla_\mu\kappa_\rho)\,. \quad (2.10)\end{aligned}$$

Then the perturbation V and $[\nabla_\mu, K]$ can be treated on the same footing. The program to move the covariant derivatives to the rightmost position is shown in Appendix C.

We denote the contribution to the trace anomaly from the m-ple X and n-ple Y insertions as $\mathcal{A}^{(mn)}$. We have to evaluate $\mathcal{A}^{(mn)}$'s, with $(m,n) = (2,0),\ (0,1)$ and $(m,n) = (4,0),\ (2,1),\ (0,2)$ in two and four space-time dimensions, respectively. X and Y involve the covariant derivative once and twice respectively, and first derivatives of $I,\ K,\ \kappa$ vanish in the coincident limit of x and x'. Therefore, moving the covariant derivatives involved in X and Y to the rightmost position, we have terms consisted of second derivatives of $I,\ K,\ \kappa$ in two space-time dimensions, while we have terms consisted of fourth derivatives of $I,\ K,\ \kappa$ or quadratic terms in second derivatives of $I,\ K,\ \kappa$ in four space-time dimensions.

The integrations over the angle variables of the momentum yield the contraction of the indices of the momentum:

$$\begin{aligned} & \int d^nk\, F(k^2)\frac{k_{\mu_1}k_{\mu_2}\cdots k_{\mu_{2m}}}{(k^2)^m} \\ &= f(m)\, g_{\mu_1\mu_2\cdots\mu_{2m}}\int d^nk\, F(k^2)\,, \quad (2.11)\end{aligned}$$

with

$$f(m) = \frac{\Gamma(\frac{n}{2})}{2^m\Gamma(\frac{n}{2}+m)}\,, \quad (2.12)$$

where the generalized metric g is the one introduced in the previous paper[5,10]. In the gauge theory, e^{-tK} consists of the transverse part $(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2})e^{tk^2}$ and the longitudinal part $\frac{k_\mu k_\nu}{k^2}e^{\frac{t}{\alpha}k^2}$. We utilize the method given in the previous paper[10] to perform the integrations over the angle variables, that is, symmetrization operation. For convenience, we denote the transverse and the longitudinal projection operators as $P^{(0)}$ and $P^{(1)}$, respectively. In order to know the symmetrized expression with j-th insertions of $P^{(1)}$, that is,

$$\begin{aligned} J_j &= \mathrm{Sym}\{P^{(\epsilon_1)}M_1P^{(\epsilon_2)}M_2\cdots M_{m-1}P^{(\epsilon_m)}\} \\ &\quad + (\,\text{Permutations with } \sum_i \epsilon_i = j\,)\,, \quad (2.13)\end{aligned}$$

we evaluate the following object,

$$\begin{aligned} J(y) &= \mathrm{Sym}\{P(y)M_1P(y)M_2P(y)\cdots M_{m-1}P(y)\} \\ &= \sum_{j=0}^{m} a_j y^j (k^2)^l\,, \quad (2.14)\end{aligned}$$

with $P(y) = g_{\mu\nu} - y\frac{k_\mu k_\nu}{k^2}$. Here k^2 dependence of J is explicitly extracted and the coefficients a_k's are independent of the momentum. Then J_k's are obtained by

$$J_0 = J(1) = \sum_{j=0}^{m} a_j(k^2)^l\,, \quad (2.15)$$

$$J_1 = -J'(1) = -\sum_{j=1}^{m} j\,a_j(k^2)^l\,, \quad (2.16)$$

$$J_2 = \frac{1}{2}J''(1) = \sum_{j=2}^{m}\frac{j(j-1)}{2}\,a_j(k^2)^l\,, \quad (2.17)$$

$$\vdots$$

$$J_m = \frac{(-1)^m}{m!}J^{(m)}(1) = (-1)^m a_m(k^2)^l\,. \quad (2.18)$$

Integration formulas over the momentum and the parameters are obtained by Mathematica.

$$\begin{aligned} p(m,2l,j) &\equiv -i(4\pi)^{\frac{n}{2}}\int\frac{d^nk}{(2\pi)^n}\int\cdots\int\prod_{i=0}^{m}dt_i \\ &\times\delta(1-\sum_{i=0}^{m}t_i)(k^2)^l\exp\{(\sum_{i=0}^{j}\frac{t_i}{\alpha}+\sum_{i=j+1}^{m}t_i)k^2\}, \quad (2.19)\end{aligned}$$

where the factor $(-i)(4\pi)^{\frac{n}{2}}$ is inserted to simplify the expression of p. This is essentially the same as pi in the previous paper[10]. The first argument of p stands for the number of parameter integration and differs by 1 from the definition of the first argument of pi. For our purposes, we need $p(m,l,j)$'s with $m=2$ through 7, $l=0$ through 8, and $j=0$ through m. Denoting a_j for the specific term with m-ple parameter integration, a-ple X and b-ple Y insertions, j-ple insertions of the longitudinal projection operator, and l-th order in k^2 as $V^{(ab)}(m,2l,j)$, we have the following recombination formulas,

$$\begin{aligned} \mathcal{A}^{(01)} &\propto \sum_{j=0}^{2}V^{(01)}(2,0,j)\sum_{k=0}^{j}(-1)^k\,{}_jC_k\,p(2,0,k) \\ &+\sum_{j=0}^{3}V^{(01)}(3,2,j)\sum_{k=0}^{j}(-1)^k\,{}_jC_k\,p(3,2,k)\,, \quad (2.20)\end{aligned}$$

$$\begin{aligned} \mathcal{A}^{(20)} &\propto \sum_{j=0}^{3}V^{(20)}(3,2,j)\sum_{k=0}^{j}(-1)^k\,{}_jC_k\,p(3,2,k) \\ &+\sum_{j=0}^{4}V^{(20)}(4,4,j)\sum_{k=0}^{j}(-1)^k\,{}_jC_k\,p(4,4,k)\,, \quad (2.21)\end{aligned}$$

$$\begin{aligned} \mathcal{A}^{(02)} &\propto \sum_{j=0}^{3}V^{(02)}(3,0,j)\sum_{k=0}^{j}(-1)^k\,{}_jC_k\,p(3,0,k) \\ &+\sum_{j=0}^{4}V^{(02)}(4,2,j)\sum_{k=0}^{j}(-1)^k\,{}_jC_k\,p(4,2,k) \\ &+\sum_{j=0}^{5}V^{(02)}(5,4,j)\sum_{k=0}^{j}(-1)^k\,{}_jC_k\,p(5,4,k)\,, \quad (2.22)\end{aligned}$$

$$\mathcal{A}^{(21)} \propto \sum_{j=0}^{4}V^{(21)}(4,2,j)\sum_{k=0}^{j}(-1)^k\,{}_jC_k\,p(4,2,k)$$

$$+\sum_{j=0}^{5} V^{(21)}(5,4,j) \sum_{k=0}^{j} (-1)^k \, {}_jC_k \, p(5,4,k)$$

$$+\sum_{j=0}^{6} V^{(21)}(6,6,j) \sum_{k=0}^{j} (-1)^k \, {}_jC_k \, p(6,6,k)\,, \quad (2.23)$$

$$\mathcal{A}^{(40)} \propto \sum_{j=0}^{5} V^{(40)}(5,4,j) \sum_{k=0}^{j} (-1)^k \, {}_jC_k \, p(5,4,k)$$

$$+\sum_{j=0}^{6} V^{(40)}(6,6,j) \sum_{k=0}^{j} (-1)^k \, {}_jC_k \, p(6,6,k)$$

$$+\sum_{j=0}^{7} V^{(40)}(7,8,j) \sum_{k=0}^{j} (-1)^k \, {}_jC_k \, p(7,8,k)\,, \quad (2.24)$$

where the proportional constant is $(-1)(4\pi)^{-\frac{n}{2}}$.

III Contraction of tensor indices of $I_{\mu\cdots}$ and $\sigma^{\ a}_{\mu\cdots}$

The coincident limits of $I_{\mu\cdots}$ and $\sigma^{a}_{\mu\cdots}$ are given in the previous paper[5]. Here we explain the program to contract the tensor indices of $I_{\mu\cdots}$ and $\sigma^{\ a}_{\mu\cdots}$.

After symmetrization, each terms consist of two $[\sigma^{\ a}_{\mu\nu\rho}]$'s(denoted by $s[\mu,\nu,\rho,a]$ in the program), or one $[\sigma^{\ a}_{\mu\nu\rho\tau\varepsilon}]$(denoted by $s[\mu,\nu,\rho,\tau,\varepsilon,a]$ in the program), or one $[\sigma^{\ a}_{\mu\nu\rho}]$ and one $[I_{\mu\nu\rho\tau}]$(denoted by $i[\mu,\nu,\rho,\tau]$), or one $[I_{\mu\nu\rho\tau\varepsilon\omega}]$(denoted by $i[\mu,\nu,\rho,\tau,\varepsilon,\omega]$). Before dissolving the generalized metric into the usual metrics with two indices, we notice the eighth order metric $g_{\mu_1\cdots\mu_8}$(denoted by $g[\mu_1,\cdots,\mu_8]$) contracted with the above tensors vanish identically on account of its totally symmetric property under exchange of Lorentz indices:

$$g[\alpha,\beta,\mu_1,\cdots,\mu_6]\,s[\mu_1,\cdots,\mu_6]=0\,, \quad (3.1)$$

$$g[\alpha,\beta,\mu_1,\cdots,\mu_6]\,s[\mu_1,\mu_2,\mu_3,\nu] \times s[\mu_4,\mu_5,\mu_6,\nu]=0\,, \quad (3.2)$$

$$g[\alpha,\mu_1,\cdots,\mu_7]\,s[\mu_1,\cdots,\mu_4] \times s[\mu_5,\mu_6,\mu_7,\beta]=0\,, \quad (3.3)$$

$$g[\mu_1,\cdots,\mu_8]\,s[\mu_1,\cdots,\mu_4] \times s[\mu_5,\cdots,\mu_8]=0\,. \quad (3.4)$$

In fact $s[\mu_1,\cdots,\mu_4]$ vanishes if any three of indices are symmetrized. Likewise $i[\mu_1,\cdots,\mu_5,\beta]$ vanishes if first 5 indices are symmetrized.

The sixth order metric $g_{\mu_1\cdots\mu_6}(g[\mu_1,\cdots,\mu_6])$ contracted with i's or some s's vanish identically:

$$g[\alpha,\mu_1,\cdots,\mu_5]\,i[\mu_1,\cdots,\mu_5,\beta]=0\,, \quad (3.5)$$

$$g[\alpha,\mu_1,\cdots,\mu_5]\,s[\beta,\mu_1,\cdots,\mu_5]=0\,, \quad (3.6)$$

$$g[\mu_1,\cdots,\mu_6]\,s[\mu_1,\cdots,\mu_6]=0\,. \quad (3.7)$$

Only surviving terms are $g[\alpha,\beta,\mu_1,\cdots,\mu_4]\,s[\mu_1,\cdots,\mu_4,\nu,\nu]$, $g[\alpha,\beta,\mu_1,\cdots,\mu_4]\,s[\mu_1,\mu_2,\nu,\rho]\,s[\mu_3,\mu_4,\nu,\rho]$, $g[\alpha,\beta,\mu_1,\cdots,\mu_4]\,s[\mu_1,\mu_2,\nu,\nu]\,s[\mu_3,\mu_4,\rho,\rho]$ and their permutations. In any case uncontracted indices α and β must be in the sixth order metric $g_{\mu_1\cdots\mu_6}$. The above three kinds of contractions are evaluated beforehand, and their results are substituted into the terms with sixth order metric.

The fourth order metric $g_{\mu_1\cdots\mu_4}$ is dissolved into the usual metric $g_{\mu\nu}$'s, and contractions of indices of s and i are performed. The program to perform symmetrization and contraction for $V^{40}(6,6,0)$ is shown as an example in Appendix D. The computation was done by a machine with memory 256MB and with a Pentium IV processor. Due to this restricted computer power, $V^{40}(6,6,n)\,(n=2,3,4)$ and $V^{40}(7,8,n)\,(n=2,3,4,5)$ could not be evaluated at once, and we divided them into partial terms, and excuted the program.

Finally, the recombination of terms as explained in Sec.2 is performed by the program shown in Appendix E.

IV Results

The trace anomalies in two space-time dimensions are obtained as[11],

$$g^{\mu\nu} < T^{(A)}_{\mu\nu} >_{anomaly}$$
$$= -\frac{1}{4\pi}\,\mathrm{tr}\big[c^{(2)}_1(\omega) g_{\mu\nu}R + c^{(2)}_2(\omega)R_{\mu\nu}\big]$$
$$= -\frac{1}{4\pi}\,(-\frac{2}{3}\,R)\,, \quad (4.1)$$

$$g^{\mu\nu} < T^{(c)}_{\mu\nu} >_{anomaly}$$
$$= -\frac{1}{4\pi}\,(-\frac{1}{3}\,R)\,, \quad (4.2)$$

with

$$c^{(2)}_1(\omega) = \frac{1}{6} + \frac{\alpha^{\omega-1}-1}{12(\omega-1)} = \frac{1}{6} + \frac{1}{12}\ln\alpha\,. \quad (4.3)$$

$$c^{(2)}_2(\omega) = -1 - \frac{\alpha^{\omega-1}-1}{6(\omega-1)} = -1 - \frac{1}{6}\ln\alpha\,. \quad (4.4)$$

Here $\omega=\frac{n}{2}$ and in two space-time dimensions we take the limit of $\omega\to 1$. The gauge dependence of $c^{(2)}_1$ and $c^{(2)}_2$ are cancelled out after taking the trace with respect to the Lorentz indices. These results (4.3) and (4.4) are consistent with Gusyuin and Gorbar's results in Ref.4.

In four space-time dimensions we have obtained the following trace anomalies [11] in the limit of $\omega\to 2$,

$$g^{\mu\nu} < T^{(A)}_{\mu\nu} >_{anomaly}$$
$$= -\frac{1}{16\pi^2}\,\mathrm{tr}\Big[c^{(4)}_1(\omega) g_{\alpha\beta}\big\{2R^2_{\mu\nu\rho\sigma} - 2R^2_{\mu\nu} + 5R^2 - 12\Box R\big\}$$
$$+c^{(4)}_2(\omega)R_{\alpha\beta}R + c^{(4)}_3(\omega)R_{\alpha\mu\nu\rho}R_{\beta}{}^{\mu\nu\rho}$$
$$+c^{(4)}_4(\omega)R_{\alpha\mu}R_{\beta}{}^{\mu} + c^{(4)}_5(\omega)R_{\alpha\mu\beta\nu}R^{\mu\nu}$$
$$+c^{(4)}_6(\omega)\Box R_{\alpha\beta} + c^{(4)}_7(\omega)\nabla_\alpha\nabla_\beta R\Big]$$

$$= -\frac{1}{16\pi^2}\Big[-\frac{11}{180}R^2_{\mu\nu\rho\sigma}+\frac{43}{90}R^2_{\mu\nu}-\frac{1}{9}R^2 +\frac{1}{60}(2-5\ln\alpha)\Box R\Big], \quad (4.5)$$

$$g^{\mu\nu} < T^{(c)}_{\mu\nu} >_{anomaly} = -\frac{1}{16\pi^2}\Big[-\frac{1}{90}R^2_{\mu\nu\rho\sigma}+\frac{1}{90}R^2_{\mu\nu}-\frac{1}{36}R^2 +\frac{1}{15}\Box R\Big], \quad (4.6)$$

with

$$c_1^{(4)}(\omega) = \frac{1}{720}\big(2+\frac{\alpha^{\omega-2}-1}{\omega-2}\big) = \frac{1}{720}(2+\ln\alpha), \quad (4.7)$$

$$c_2^{(4)}(\omega) = -\frac{1}{36}\big(6+\frac{\alpha^{\omega-2}-1}{\omega-2}\big) = -\frac{1}{36}(6+\ln\alpha), \quad (4.8)$$

$$c_3^{(4)}(\omega) = -\frac{1}{180}\big\{15+\frac{2(\alpha^{\omega-2}-1)}{\omega-2}\big\} = -\frac{1}{180}(15+2\ln\alpha), \quad (4.9)$$

$$c_4^{(4)}(\omega) = \frac{1}{90}\big\{45+\frac{2(\alpha^{\omega-2}-1)}{\omega-2}\big\} = \frac{1}{90}(45+2\ln\alpha), \quad (4.10)$$

$$c_5^{(4)}(\omega) = \frac{1}{90}\frac{\alpha^{\omega-2}-1}{\omega-2} = \frac{1}{90}\ln\alpha, \quad (4.11)$$

$$c_6^{(4)}(\omega) = \frac{1}{60}\big(10+\frac{\alpha^{\omega-2}-1}{\omega-2}\big) = \frac{1}{60}(10+\ln\alpha), \quad (4.12)$$

$$c_7^{(4)}(\omega) = -\frac{1}{30}\frac{\alpha^{\omega-2}-1}{\omega-2} = -\frac{1}{30}\ln\alpha, \quad (4.13)$$

The gauge dependent trace anomaly in four space-time dimensions, (4.5) agrees with Endo's one in Ref.3. The gauge dependent coefficients before taking the trace with respect to the Lorentz indices, (4.7) through (4.13) are new results. It is said that the ζ function regularization and the dimensional regularization give different results for the coefficient of $\Box R$ [12,3] in the trace anomaly. In our computation, we regularized the ultraviolet divergence by cutting off the integration region around $t=0$. In the Feynman gauge, $\alpha=1$, our result agrees with the one of the ζ function regularization.

Appendix A. Definition of bivector $I^{\alpha}{}_{\beta}(x,x')$ and the coincidence limits of its derivatives

The geodesic parallel displacement bivector operator $I^{\alpha}{}_{\beta}(x,x')$ is defined like the bispinor opertor, whose properties are summarized in the Appendix C of the previous paper [5]. The properties of $I^{\alpha}{}_{\beta}(x,x')$ can be read from those of the bispinor, and we list them up for reader's convenience.

$$g^{\mu\nu}\,\nabla_\mu I^{\alpha}{}_{\beta}(x,x')\,\nabla_\nu\sigma(x,x') = 0\,, \quad (A.1)$$

and the boundary condition,

$$I^{\alpha}{}_{\beta}(x,x) = \delta^{\alpha}{}_{\beta}\,. \quad (A.2)$$

We use the following notations for derivatives of $I^{\alpha}{}_{\beta}(x,x')$ and their coincidence limits,

$$I_{\mu_1\mu_2\cdots}{}^{\alpha}{}_{\beta}(x,x') = \nabla_{\mu_1}\nabla_{\mu_2}\cdots I^{\alpha}{}_{\beta}(x,x')\,, \quad (A.3)$$

$$[\,I_{\mu_1\mu_2\cdots}{}^{\alpha}{}_{\beta}(x)\,] = \lim_{x'\to x} I_{\mu_1\mu_2\cdots}{}^{\alpha}{}_{\beta}(x,x')\,. \quad (A.4)$$

The commutation of two covariant derivatives gives the following terms,

$$[\,\nabla_\mu,\,\nabla_\nu\,]\,I_{\rho_1\cdots\rho_m}{}^{\alpha}{}_{\beta}(x,x') = R_{\mu\nu}{}^{\alpha}{}_{\gamma}\,I_{\rho_1\cdots\rho_m}{}^{\gamma}{}_{\beta}(x,x') - \sum_{j=1}^{m} R_{\mu\nu}{}^{\tau}{}_{\rho_j}\,I_{\rho_1\cdots\tau\cdots\rho_m}{}^{\alpha}{}_{\beta}(x,x')\,. \quad (A.5)$$

Taking the covariant derivatives of the orthonormal condition (A.1) and making use of the relations (A.5), we obtain the following expressions,

$$[\,I_\mu{}^{\alpha}{}_{\beta}\,] = 0\,, \quad (A.6)$$

$$[\,I_{\mu\nu}{}^{\alpha}{}_{\beta}\,] = \frac{1}{2}\,R_{\mu\nu}{}^{\alpha}{}_{\beta}\,, \quad (A.7)$$

$$[\,I_{\mu\nu\rho}{}^{\alpha}{}_{\beta}\,] = \frac{1}{3}\,\nabla_{(\mu}R_{\nu)\rho}{}^{\alpha}{}_{\beta}\,, \quad (A.8)$$

$$\begin{aligned}[\,I_{\mu\nu\rho\tau}{}^{\alpha}{}_{\beta}\,] = {}& \frac{1}{8}\big(R_{\mu\nu}{}^{\alpha}{}_{\gamma}R_{\rho\tau}{}^{\gamma}{}_{\beta} + R_{(\mu\rho}{}^{\alpha}{}_{\gamma}R_{\nu)\tau}{}^{\gamma}{}_{\beta} + R_{(\mu\tau}{}^{\alpha}{}_{\gamma}R_{\nu)\rho}{}^{\gamma}{}_{\beta} + R_{\rho\tau}{}^{\alpha}{}_{\gamma}R_{\mu\nu}{}^{\gamma}{}_{\beta}\big) \\ & + \frac{1}{4}\big(\nabla_\mu\nabla_\nu R_{\rho\tau}{}^{\alpha}{}_{\beta} + \nabla_{(\mu}\nabla_\rho R_{\nu)\tau}{}^{\alpha}{}_{\beta}\big) \\ & + \frac{1}{12}g^{\gamma\delta}\big(R_{\mu\gamma}{}^{\alpha}{}_{\beta}R_{\nu(\rho\tau)\delta} + R_{\nu\gamma}{}^{\alpha}{}_{\beta}R_{\mu(\rho\tau)\delta} + R_{\rho\alpha}{}^{\alpha}{}_{\beta}R_{\mu(\nu\tau)\delta} + R_{\tau\alpha}{}^{\alpha}{}_{\beta}R_{\nu(\mu\rho)\delta}\big)\,. \end{aligned} \quad (A.9)$$

If we contract two pairs of indices of $[I_{\mu\nu\rho\tau}{}^{\alpha}{}_{\beta}]$, we obtain the following formulas,

$$g^{\mu\nu}g^{\rho\tau}[I_{\mu\nu\rho\tau}{}^{\alpha}{}_{\beta}] = -g^{\mu\tau}g^{\nu\rho}[I_{\mu\nu\rho\tau}{}^{\alpha}{}_{\beta}] = \frac{1}{2}g^{\mu\rho}g^{\nu\tau}R_{\mu\nu}{}^{\alpha}{}_{\gamma}R_{\rho\tau}{}^{\gamma}{}_{\beta}\,, \quad (A.10)$$

$$g^{\mu\rho}g^{\nu\tau}[I_{\mu\nu\rho\tau}{}^{\alpha}{}_{\beta}] = 0\,, \quad (A.11)$$

$$g^{\mu\nu}[I_{\mu\nu\rho\alpha}{}^{\rho}{}_{\beta}] = \frac{1}{3}R_{\alpha\mu\beta\nu}R^{\mu\nu} - \frac{5}{8}R_{\alpha\mu\nu\rho}R_{\beta}{}^{\mu\nu\rho} - \frac{3}{4}\Box R_{\alpha\beta} + \frac{1}{4}\nabla_\alpha\nabla_\beta R - \frac{5}{12}R_{\alpha\mu}R_{\beta}{}^{\mu}\,, \quad (A.12)$$

$$g^{\mu\nu}[I_{\mu\rho\nu\alpha}{}^{\rho}{}_{\beta}] = -\frac{1}{6}R_{\alpha\mu\beta\nu}R^{\mu\nu} - \frac{3}{8}R_{\alpha\mu\nu\rho}R_{\beta}{}^{\mu\nu\rho} - \frac{3}{4}\Box R_{\alpha\beta} + \frac{1}{4}\nabla_\alpha\nabla_\beta R - \frac{5}{12}R_{\alpha\mu}R_{\beta}{}^{\mu}\,, \quad (A.13)$$

$$\begin{aligned} g^{\mu\nu}[I_{\rho\mu\nu\alpha}{}^{\rho}{}_{\beta}] &= -\frac{1}{6}R_{\alpha\mu\beta\nu}R^{\mu\nu} - \frac{1}{8}R_{\alpha\mu\nu\rho}R_{\beta}{}^{\mu\nu\rho} \\ &-\frac{3}{4}\Box R_{\alpha\beta} + \frac{1}{4}\nabla_\alpha\nabla_\beta R - \frac{5}{12}R_{\alpha\mu}R_{\beta}{}^{\mu}\,, \qquad (A.14) \\ g^{\mu\nu}[I_{\mu\nu\alpha\rho}{}^{\rho}{}_{\beta}] &= \frac{1}{3}R_{\alpha\mu\beta\nu}R^{\mu\nu} - \frac{5}{8}R_{\alpha\mu\nu\rho}R_{\beta}{}^{\mu\nu\rho} \\ &+\frac{1}{4}\Box R_{\alpha\beta} + \frac{1}{4}\nabla_\alpha\nabla_\beta R - \frac{5}{12}R_{\alpha\mu}R_{\beta}{}^{\mu}\,, \qquad (A.15) \\ g^{\mu\nu}[I_{\mu\rho\alpha\nu}{}^{\rho}{}_{\beta}] &= -\frac{1}{6}R_{\alpha\mu\beta\nu}R^{\mu\nu} + \frac{7}{8}R_{\alpha\mu\nu\rho}R_{\beta}{}^{\mu\nu\rho} \\ &+\frac{1}{4}\Box R_{\alpha\beta} - \frac{1}{4}\nabla_\alpha\nabla_\beta R + \frac{7}{12}R_{\alpha\mu}R_{\beta}{}^{\mu}\,, \qquad (A.16) \\ g^{\mu\nu}[I_{\rho\mu\alpha\nu}{}^{\rho}{}_{\beta}] &= -\frac{1}{6}R_{\alpha\mu\beta\nu}R^{\mu\nu} + \frac{5}{8}R_{\alpha\mu\nu\rho}R_{\beta}{}^{\mu\nu\rho} \\ &+\frac{1}{4}\Box R_{\alpha\beta} - \frac{1}{4}\nabla_\alpha\nabla_\beta R + \frac{7}{12}R_{\alpha\mu}R_{\beta}{}^{\mu}\,, \qquad (A.17) \\ g^{\mu\nu}[I_{\mu\alpha\nu\rho}{}^{\rho}{}_{\beta}] &= \frac{1}{12}R_{\alpha\mu\beta\nu}R^{\mu\nu} - \frac{3}{8}R_{\alpha\mu\nu\rho}R_{\beta}{}^{\mu\nu\rho} \\ &+\frac{1}{4}\Box R_{\alpha\beta} + \frac{1}{4}\nabla_\alpha\nabla_\beta R - \frac{5}{12}R_{\alpha\mu}R_{\beta}{}^{\mu}\,, \qquad (A.18) \\ g^{\mu\nu}[I_{\mu\alpha\rho\nu}{}^{\rho}{}_{\beta}] &= -\frac{1}{6}R_{\alpha\mu\beta\nu}R^{\mu\nu} + \frac{3}{8}R_{\alpha\mu\nu\rho}R_{\beta}{}^{\mu\nu\rho} \\ &+\frac{1}{4}\Box R_{\alpha\beta} - \frac{1}{4}\nabla_\alpha\nabla_\beta R + \frac{7}{12}R_{\alpha\mu}R_{\beta}{}^{\mu}\,, \qquad (A.19) \\ g^{\mu\nu}[I_{\rho\alpha\mu\nu}{}^{\rho}{}_{\beta}] &= \frac{1}{12}R_{\alpha\mu\beta\nu}R^{\mu\nu} + \frac{1}{8}R_{\alpha\mu\nu\rho}R_{\beta}{}^{\mu\nu\rho} \\ &+\frac{1}{4}\Box R_{\alpha\beta} - \frac{1}{4}\nabla_\alpha\nabla_\beta R + \frac{1}{12}R_{\alpha\mu}R_{\beta}{}^{\mu}\,, \qquad (A.20) \\ g^{\mu\nu}[I_{\alpha\mu\nu\rho}{}^{\rho}{}_{\beta}] &= -\frac{1}{6}R_{\alpha\mu\beta\nu}R^{\mu\nu} - \frac{1}{8}R_{\alpha\mu\nu\rho}R_{\beta}{}^{\mu\nu\rho} \\ &+\frac{1}{4}\Box R_{\alpha\beta} + \frac{1}{4}\nabla_\alpha\nabla_\beta R + \frac{1}{12}R_{\alpha\mu}R_{\beta}{}^{\mu}\,, \qquad (A.21) \\ g^{\mu\nu}[I_{\alpha\mu\rho\nu}{}^{\rho}{}_{\beta}] &= g^{\mu\nu}[I_{\alpha\rho\mu\nu}{}^{\rho}{}_{\beta}] \\ &= g^{\mu\nu}[I_{\rho\alpha\mu\nu}{}^{\rho}{}_{\beta}]\,. \qquad (A.22) \end{aligned}$$

In deriving these formulas, we utilized the Bianchi identity and its consequences,

$$\nabla_\mu R_{\nu\rho\alpha\beta} + \nabla_\nu R_{\rho\mu\alpha\beta} + \nabla_\rho R_{\mu\nu\alpha\beta} = 0\,, \qquad (A.23)$$

$$\begin{aligned} \nabla_\alpha\nabla^\mu R_{\beta\mu} &= \nabla_\beta\nabla^\mu R_{\alpha\mu} = \frac{1}{2}\nabla_\alpha\nabla_\beta R\,, \qquad (A.24) \\ \nabla^\mu\nabla^\nu R_{\alpha\mu\nu\beta} &= \Box R_{\alpha\beta} - \nabla^\mu\nabla_\alpha R_{\mu\beta}\,, \qquad (A.25) \\ \nabla^\mu\nabla_\alpha R_{\mu\beta} &= \nabla^\mu\nabla_\beta R_{\mu\alpha} = \frac{1}{2}\nabla_\alpha\nabla_\beta R \\ &-R_{\mu\alpha}R^{\mu}{}_{\beta} - R_{\alpha\mu\beta\nu}R^{\mu\nu}\,. \qquad (A.26) \end{aligned}$$

Appendix B. Derivation of Eq.(1.10)

The propagator of the gauge field satisfies the following equation in the general covariant gauge,

$$\begin{aligned} &\big(\delta^\mu{}_\nu\Box - \nabla_\nu\nabla^\mu + \frac{1}{\alpha}\nabla^\mu\nabla_\nu\big) \\ &\times < T[\,A^\nu(x)A_\rho(y)\,] >= \frac{i}{\sqrt{-g}}\,\delta^\mu{}_\rho\,\delta^n(x-y) \qquad (B.1) \end{aligned}$$

The gauge field itself satisfies the homogeneous equation. The delta function in the righthand side of (B.1) originates from the following facts,

$$\begin{aligned} &\nabla_0 < T[\,\nabla^0 A^i(x)\,A_\rho(y)\,] > \\ &= \delta(x^0-y^0) < [\,\nabla^0 A^i(x),\,A_\rho(y)\,] > \\ &\quad + < T[\,\nabla_0\nabla^0 A^i(x)\,A_\rho(y)\,] > \\ &= \frac{i}{\sqrt{-g}}\,\delta^i{}_\rho\,\delta^n(x-y) + \cdots\,, \qquad (B.2) \\ &g^{00}\nabla_0 < T[\,\frac{1}{\alpha}\nabla_\nu A^\nu(x)\,A_\rho(y)\,] > \\ &= \delta(x^0-y^0) < \frac{1}{\alpha}g^{00}[\,\nabla_\nu A^\nu(x),\,A_\rho(y)\,] > \\ &\quad +g^{00} < T[\,\frac{1}{\alpha}\nabla_0\nabla_\nu A^\nu(x)\,A_\rho(y)\,] > \\ &= \frac{i}{\sqrt{-g}}\,\delta^0{}_\rho\,\delta^n(x-y) + \cdots\,. \qquad (B.3) \end{aligned}$$

In the Euclidean space we denote eigenvalues and eigenfunctions of the operator $Y^{(A)}$ as

$$\big(\delta^\mu{}_\nu\Box - \nabla_\nu\nabla^\mu + \frac{1}{\alpha}\nabla^\mu\nabla_\nu\big)u_k^\nu(x) = \lambda_k u_k^\mu(x)\,. \qquad (B.4)$$

Then the propagator can be expressed as

$$< T[\,A^\mu(x)\,A_\nu(y)\,] >= \sum_k \frac{i}{\lambda_k}u_k^\mu(x)u_{\nu,k}^*(y)\,. \qquad (B.5)$$

Equation (B.1) is reproduced due to the completeness condition of the eigenfunctions,

$$\sum_k u_k^\mu(x)u_{\nu,k}^*(y) = \frac{1}{\sqrt{-g}}\,\delta^\mu{}_\nu\delta^n(x-y)\,. \qquad (B.6)$$

Now we take the variation of (1.3) under local Weyl transformation, $\delta g^{\mu\nu} = 2\alpha(x)g^{\mu\nu}$.

$$\begin{aligned} &\delta W^{(A)}[g^{\mu\nu}] \\ &= \frac{i}{2}\int_\epsilon^\infty dt\,\mathrm{Tr}\Big[\,\delta Y^{(A)}e^{-tY^{(A)}}\,\Big] \\ &= \frac{i}{2}\int d^nx\sqrt{-g}\lim_{x'\to x}\int_\epsilon^\infty dt\,\big(\delta Y^{(A)}e^{-tY^{(A)}}\big)^\mu{}_\nu \\ &\quad \times\sum_k u_k^\nu(x)u_{\mu,k}^*(x') \\ &= \frac{i}{2}\int d^nx\sqrt{-g}\sum_k u_{\mu,k}^*(x)\int_\epsilon^\infty dt\,e^{-t\lambda_k}\big(\delta Y^{(A)}\big)^\mu{}_\nu u_k^\nu(x) \\ &= i\int d^nx\sqrt{-g}\alpha(x)\sum_k u_{\mu,k}^*(x)u_k^\mu(x)\int_\epsilon^\infty dt\,e^{-t\lambda_k}\lambda_k \\ &\quad +i\int d^nx\sqrt{-g}\nabla^\mu\alpha(x)\sum_k\frac{1}{\lambda_k}\Big\{(\nabla_\mu u_{\nu,k}^* \\ &\quad -\nabla_\nu u_{\mu,k}^*)u_k^\nu - \frac{n-2}{2}u_k^{*\nu}(\nabla_\mu u_{\nu,k} - \nabla_\nu u_{\mu,k}) \\ &\quad +\frac{n}{2\alpha}\nabla_\nu u_k^{*\nu}u_{\mu,k}\Big\} \\ &= i\int d^nx\sqrt{-g}\alpha(x)\sum_k u_{\mu,k}^*(x)u_k^\nu(x)\,e^{-\epsilon\lambda_k} \\ &\quad +i\int d^nx\sqrt{-g}\alpha(x)\sum_k\frac{1}{\lambda_k}\Big\{\frac{n-4}{2}\nabla^\mu u_k^{*\nu} \end{aligned}$$

$$\times(\nabla_\mu u_{\nu,k} - \nabla_\nu u_{\mu,k}) - \frac{n}{2\alpha}\nabla_\nu u_k^{*\nu}\nabla_\mu u_k^\mu$$
$$-(\Box u_{\nu,k}^* - \nabla_\mu\nabla_\nu u_k^{*\mu})u_k^\nu + \frac{n-2}{2}u_k^{*\nu}(\Box u_{\nu,k}$$
$$-\nabla_\mu\nabla_\nu u_k^\mu) - \frac{n}{2\alpha}\nabla_\mu\nabla_\nu u_k^{*\nu}u_k^\mu\Big\}. \quad \text{(B.7)}$$

The first term in the last line of (B.7) is the trace anomaly, and the other terms should become the canonical trace terms, that is,

$$\int d^n x\sqrt{-g}\alpha(x) < \frac{n-4}{4}F_{\mu\nu}^2 - \frac{n-2}{\alpha}A^\mu\nabla_\mu\nabla_\nu A^\nu$$
$$-\frac{n}{2\alpha}(\nabla_\mu A^\mu)^2 > . \quad \text{(B.8)}$$

The first and the second terms in the curly bracket in the last line of (B.7) correspond to the first and the third term of the expression (B.8) except for the contact terms proportional to $\delta^n(0)$. Making use of Eq.(B.4), the third, the fourth, and the fifth terms in the curly bracket in the last line of (B.7) are rewritten as,

$$i\int d^n x\sqrt{-g}\alpha(x)\sum_k\frac{1}{\lambda_k}\left[\frac{n-4}{2}\lambda_k u_{\mu,k}^* u_k^\mu\right.$$
$$\left.-\frac{n-2}{2\alpha}\{(\nabla_\nu\nabla_\mu u_k^{*\mu})u_k^\nu + u_k^{*\nu}(\nabla_\nu\nabla_\mu u_k^\mu)\}\right]. \quad \text{(B.9)}$$

The last two terms of (B.9) correspond to the second term of (B.8) except for the contact term. The first term in the curly bracket of (B.9) becomes the contact term,

$$i\frac{n(n-4)}{2}\,\delta^n(0)\int d^n x\sqrt{-g}\alpha(x)\,. \quad \text{(B.10)}$$

On the other hand, if we move the time derivatives in the expression (B.8) to the outside of the time-ordered product, there appear the following contact terms,

$$i\Big\{\frac{(n-4)(n-1)}{2} + (n-2) - \frac{n}{2}\Big\}\,\delta^n(0)$$
$$\times\int d^n x\sqrt{-g}\alpha(x). \quad \text{(B.11)}$$

The three terms in the curly bracket sum up to $\frac{n(n-4)}{2}$. Thus we have finally proved the gauge field part of (1.10). The ghost field part is much easilly proved, though we do not show the proof here.

Appendix C. Application of the Leipnitz rule for the covariant derivative

```
(* products of differential operators and
 application of derivative *)
SetAttributes[DP,Flat]
(* d[m] and p[m] stand for \nabla_\mu and
 \kappa_\mu, respectively. *)
DP[x__,y1_+y2_,z__]:= DP[x,y1,z] + DP[x,y2,z]
DP[x__,d[m_],e,y__]:=
  DP[x,e,d[m],y] + DP[x,e,a[m],e,y]
DP[x__,d[m_],p[n__],y__]:=
  DP[x,p[n],d[m],y] + DP[x,p[m,n],y]
DP[x__,d[m_],a[n__],y__]:=
  DP[x,a[n],d[m],y] + DP[x,a[m,n],y]
DP[x__,d[m_],h[n__],y__]:= DP[x,h[n],d[m],y]
DP[x__,d[m_],z[n__]]:= DP[x,z[m,n]]
 (* The last argument of z is redundunt. *)

(* definition of basic operators *)
x[m_,n_]:=
  DP[h[m,n],d[m],p[n]] + DP[h[m,n],p[m],d[n]]
y[m_,n_]:= DP[h[m,n],d[m],d[n]]

(* the heat kernel coefficient a1 *)
r20= (-1)*DP[e,x[m1,m2],e,x[m3,m4],e,z[m5]];
r01= (-1)*DP[e,y[m1,m2],e,z[m3]];

(* the heat kernel coefficient a2 *)
r40= DP[e,x[m1,m2],e,x[m3,m4],e,x[m5,m6],e,
  x[m7,m8],e,z[m9]];
r21= DP[e,x[m1,m2],e,x[m3,m4],e,y[m5,m6],e,
  z[m7]] + DP[e,x[m1,m2],e,y[m3,m4],e,
  x[m5,m6],e,z[m7]] + DP[e,y[m1,m2],e,
  x[m3,m4],e,x[m5,m6],e,z[m7]];
r02= DP[e,y[m1,m2],e,y[m3,m4],e,z[m5]];

subst1="{DP[x__,a[m_],y__]->0,
  DP[x__,p[m_,n_],y__]->0}";
subst2="{DP[x__,a[m_],y__]->0,
  DP[x__,p[m_,n_],y__]->0,
  DP[x__,z[m_,n_]]->0}";
s20=r20//. ToExpression[subst1];
s01=r01//. ToExpression[subst1];
s40=r40//. ToExpression[subst2];
s21=r21//. ToExpression[subst2];
s02=r02//. ToExpression[subst2];

SetDirectory["d:\\trace_anomaly"]
Save["derivative_r.txt",s20,s01,s40,s21,s02]
```

Appendix D. Symmetrization and contraction of tensor indices

```
$RecursionLimit=4096
SetDirectory["d:\\trace_anomaly"]

(* ny stands for the order with respect to y *)
ny=0

(* generalized metric tensors *)
SetAttributes[g,Orderless]
g[m_,m_]:=2*w

Get["derivative_r.txt"]

(* picking up terms with n-th e *)
filter[x_+y_,n_]:=filter[x,n]+filter[y,n]
```

```
filter[DP[x__],n_]:=
  If[Count[{x},e]===n,1,0]*DP[x]

v1[6]=filter[s40,6];
Print["v1 finished!"]
Clear[s40,s21,s02,s20,s01];

(* substitution into p[m,,...] *)
v2[6]=v1[6]//. {
  DP[x__,p[m_],y__]->q[m[0]]*DP[x,y],
  DP[x__,p[n1_,n2_,m_],y__]->
  -s[n1[0],n2[0],m[0],m[4]]*q[m[4]]*DP[x,y],
  DP[x__,p[n1_,n2_,n3_,n4_,m_],y__]->
  -s[n1[0],n2[0],n3[0],n4[0],m[0],m[4]]*q[m[4]]
  *DP[x,y]};
Print["v2 finished!"]
Clear[v1];

(* multiplication of tensors with explicit
    indices *)
DP[x_]:=f[x,1];
DP[x__,y_]:=DP[x]*f[y,Length[{x}]+1];
temp0=v2[6];
Print["temp0 finished!"]
Clear[v2];

(* a and b are uncontracted indices *)
l[1]=a;
(* substitution into e *)
f[e,k_]:=g[l[k],l[k+1]]-y*q[l[k]]*q[l[k+1]];

(* substitution into z[m,...] *)
f[z[m_],k_]:=g[l[k],b];
f[z[m1_,m2_,m3_],k_]:=
  r[m1[0],m2[0],l[k],b]/2;
f[z[m1_,m2_,m3_,m4_,m5_],k_]:=
  i[m1[0],m2[0],m3[0],m4[0],l[k],b];

temp1=Coefficient[Expand[temp0],y,ny];
Print["temp1 finished!"]
Clear[temp0];

(* contraction of g *)
temp2=temp1//. {
  g[x_,y_]*g[x_,z_]->g[y,z],
  g[x_,y_]*q[x_]->q[y]};
Print["temp2 finished!"]
Clear[temp1];

(* substitution into h[m,n] *)
f[h[m_,n_],k_]:=
g[m[0],n[0]]*g[l[k],l[k+1]]-
g[m[0],l[k+1]]*g[n[0],l[k]]+
g[m[0],l[k]]*g[n[0],l[k+1]]/a;

temp3=Expand[temp2]//. {
  g[x_,y_]*g[x_,z_]->g[y,z],
  g[x_,y_]*q[x_]->q[y]};
Print["temp3 finished!"]
Clear[temp2];

(* substitution into a[m,...] *)
(* c=1-1/a, a stands for the gauge parameter *)
f[a[m_,n_],k_]:=
  -2*s[m[0],n[0],n[1],n[2]]*q[n[1]]*q[n[2]]*
  g[l[k],l[k+1]]+
  c*s[m[0],n[0],l[k],n[1]]*q[n[1]]*q[l[k+1]]+
  c*s[m[0],n[0],l[k+1],n[1]]*q[l[k]]*q[n[1]];
f[a[m1_,m2_,m3_,m4_],k_]:=2*g[l[k],l[k+1]]*(-
  s[m1[0],m2[0],m3[0],m4[0],m4[1],m4[2]]*
  q[m4[1]]*q[m4[2]]+s[m1[0],m2[0],m4[1],m4[2]]*
  s[m3[0],m4[0],m4[1],m4[3]]*q[m4[2]]*q[m4[3]]+
  s[m1[0],m3[0],m4[1],m4[2]]*s[m2[0],m4[0],
  m4[1],m4[3]]*q[m4[2]]*q[m4[3]]+
  s[m1[0],m4[0],m4[1],m4[2]]*s[m2[0],m3[0],
  m4[1],m4[3]]*q[m4[2]]*q[m4[3]])+
  c*(s[m1[0],m2[0],m3[0],m4[0],l[k],m4[1]]*
  q[l[k+1]]*q[m4[1]]+ s[m1[0],m2[0],m3[0],
  m4[0],l[k+1],m4[1]]*q[l[k]]*q[m4[1]]-
  s[m1[0],m2[0],l[k],m4[1]]*s[m3[0],m4[0],
  l[k+1],m4[2]]*q[m4[1]]*q[m4[2]]-
  s[m1[0],m3[0],l[k],m4[1]]*s[m2[0],m4[0],
  l[k+1],m4[2]]*q[m4[1]]*q[m4[2]]-
  s[m1[0],m4[0],l[k],m4[1]]*s[m2[0],m3[0],
  l[k+1],m4[2]]*q[m4[1]]*q[m4[2]]-
  s[m2[0],m3[0],l[k],m4[1]]*s[m1[0],m4[0],
  l[k+1],m4[2]]*q[m4[1]]*q[m4[2]]-
  s[m2[0],m4[0],l[k],m4[1]]*s[m1[0],m3[0],
  l[k+1],m4[2]]*q[m4[1]]*q[m4[2]]-
  s[m3[0],m4[0],l[k],m4[1]]*s[m1[0],m2[0],
  l[k+1],m4[2]]*q[m4[1]]*q[m4[2]]);

v3[6,6,ny]=Expand[temp3];
Print["v3 finished!"]
Clear[temp3];

temp4=v3[6,6,ny]//. {
  g[x_,y_]^2->2*w,
  g[x_,y_]*g[x_,z_]->g[y,z]};
Print["temp4 finished!"]
Clear[v3];

temp5=temp4//. g[m_,n_]*q[m_]->q[n];
Print["temp5 finished!"]
Clear[temp4];

temp6=temp5//. {
  g[x_,y_]*s[z1___,x_,z2___]->s[z1,y,z2],
  g[x_,y_]*r[z1___,x_,z2___]->r[z1,y,z2]};
Print["temp6 finished!"]
Clear[temp5];

v4[6,6,ny]=temp6//. {g[a,b]->mt,q[n_]^2->1};
Print["v4 finished!"]
Clear[temp6];
```

```
(* symmetrization *)
temp7=v4[6,6,ny]//. q[n_]->t*o[n];
Print["temp7 finished!"]
Clear[v4];

Do[p1[nt]=Coefficient[temp7,t,nt],
  {nt,0,6,2}];
Print["p1 finished!"]
Clear[temp7];

(* no momentum to be symmetrized *)
p2[0]=p1[0];

(* quadratic terms with respect to
   the momentum to be symmetrized *)
p2[2]=p1[2]//.
  o[n1_]*o[n2_]->g[n1,n2]/(2*w);
Print["p2[2] finished!"]

(* 4-th order terms with respect to
   the momentum to be symmetrized *)
p2[4]=p1[4]//.
  o[n1_]*o[n2_]*o[n3_]*o[n4_]->
  g[n1,n2,n3,n4]/(4*w*(w+1));
Print["p2[4] finished!"]

(* 6-th order terms *)
temp8=p1[6]//. r[x__]->0;
Print["temp8 finished!"]

temp9=Coefficient[temp8,o[a]*o[b],1];
Print["temp9 finished!"]
Clear[temp8];

temp0=temp9//.
  o[n1_]*o[n2_]*o[n3_]*o[n4_]->
  1/(8*w*(w+1)*(w+2));
Print["p2[6] finished!"]
Clear[temp9];

(* contraction like g[l1,l2,l3,l4,a,b]
   s[l1,l2,m,n]s[l3,l4,m,n] *)
(* expressions of t1, t2 and t3 are
   substituted in the the last part
   of the program *)
temp1=temp0//. {
  s[x_,z1__,y_]*s[x_,z2__,y_]->t3,
  s[x_,z1__,y_]*s[y_,z2__,x_]->t3,
  s[x_,z1__,y_]*s[z2_,x_,y_,z3_]->t3,
  s[x_,z1__,y_]*s[z2_,y_,x_,z3_]->t3,
  s[z1_,x_,y_,z2_]*s[z3_,x_,y_,z4_]->t3,
  s[z1_,x_,y_,z2_]*s[z3_,y_,x_,z4_]->t3};
Clear[temp0];

temp2=temp1//. {
  s[x_,z1__,y_]*
  s[z2___,x_,z3___,y_,z4___]->t2,
  s[x_,z1__,y_]*
  s[z2___,y_,z3___,x_,z4___]->t2,
  s[z1_,x_,y_,z2_]*
  s[z3___,x_,z4___,y_,z5___]->t2,
  s[z1_,x_,y_,z2_]*
  s[z3___,y_,z4___,x_,z5___]->t2};
Clear[temp1];

temp3=temp2//. {
  s[z1___,x_,z2___,y_,z3___]*
  s[z4___,x_,z5___,y_,z6___]->t1,
  s[z1___,x_,z2___,y_,z3___]*
  s[z4___,y_,z5___,x_,z6___]->t1};
Clear[temp2];

(* contraction like g[l1,l2,l3,l4,a,b]
   s[l1,l2,l3,l4,m,m] *)
(* expressions of s1, ... , s8 are
   substituted in the the last part
   of the program *)
temp4=temp3//. {
  s[m1_,m2_,m3_,m4_,x_,x_]->s1,
  s[m1_,m2_,m3_,x_,m4_,x_]->s1,
  s[m1_,m2_,x_,m3_,m4_,x_]->s2,
  s[m1_,x_,m2_,m3_,m4_,x_]->s3,
  s[x_,m1_,m2_,m3_,m4_,x_]->s3,
  s[m1_,m2_,m3_,x_,x_,m4_]->s4,
  s[m1_,m2_,x_,m3_,x_,m4_]->s5,
  s[m1_,x_,m2_,m3_,x_,m4_]->s6,
  s[x_,m1_,m2_,m3_,x_,m4_]->s7,
  s[m1_,m2_,x_,x_,m3_,m4_]->s5,
  s[m1_,x_,m2_,x_,m3_,m4_]->s6,
  s[x_,m1_,m2_,x_,m3_,m4_]->s7,
  s[m1_,x_,x_,m2_,m3_,m4_]->s8,
  s[x_,m1_,x_,m2_,m3_,m4_]->s6,
  s[x_,x_,m1_,m2_,m3_,m4_]->s5};
Clear[temp3];

(* contraction like g[l1,l2,l3,l4,a,b]
   s[l1,l2,m,m]s[l3,l4,n,n] *)
(* ricci stands for the Ricci tensor
   and appears as ricci^2 which means
   g[l1,l2,l3,l4,a,b]r[l1,l2]r[l3,l4] *)
temp5=temp4//. {
  s[x_,x_,y_,z_]->ricci/3,
  s[x_,y_,x_,z_]->ricci/3,
  s[x_,y_,z_,x_]->-2*ricci/3,
  s[y_,x_,x_,z_]->-2*ricci/3,
  s[y_,x_,z_,x_]->ricci/3,
  s[y_,z_,x_,x_]->ricci/3};
Clear[temp4];

sub6=temp5//. s[z__]->0;
Print["sub6 finished!"]
Clear[temp5];

(* sum up to 4th order *)
v5[6,6,ny]=Sum[p2[nt],{nt,0,4,2}];
Print["v5 finished!"]
```

```
Clear[p1,p2];

temp6=Expand[v5[6,6,ny]]//. {
  g[x_,y_]^2->2*w,
  g[x_,y_]*g[x_,z_]->g[y,z]};
Clear[v5];

v6[6,6,ny]=temp6//. {
  g[x_,y_]*s[z1___,x_,z2___]->s[z1,y,z2],
  g[x_,y_]*r[z1___,x_,z2___]->r[z1,y,z2]};
Print["v6 finished!"]
Clear[temp6];

v7[6,6,ny]=v6[6,6,ny]//. g[a,b]->mt;
Print["v7 finished!"]
Clear[v6];

(* reduction of the generalized metric tensor
   to the ordinary metric tensors *)
g[m1_,m2_,m3_,m4_]:=
  g[m1,m2]*g[m3,m4]+g[m1,m3]*g[m2,m4]+
  g[m1,m4]*g[m2,m3];

v8[6,6,ny]=Expand[v7[6,6,ny]]//. {
  g[x_,y_]*s[z1___,x_,z2___]->s[z1,y,z2],
  g[x_,y_]*r[z1___,x_,z2___]->r[z1,y,z2]};
Print["v8 finished!"]
Clear[v7];

(* sc stands for the scalar curvature R *)
v9[6,6,ny]=v8[6,6,ny]//. {g[a,b]->mt,
  s[m_,m_,n_,n_]->sc/3,
  s[m_,n_,m_,n_]->sc/3,
  s[m_,n_,n_,m_]->-2*sc/3};
Print["v9 finished!"]
Clear[v8];

(* contraction of indices of i *)
(* rst, cst, crt, ddr1, and ddr2 stand for
   R_{\alpha\mu}R_{\beta}^{}{\mu},
   R_{\alpha\mu\nu\rho}R_{\beta}{}^{\mu\nu\rho},
   R_{\alpha\mu\beta\nu}R^{\mu\nu},
   \Box R_{\alpha\beta}, and
   \nabla_{\alpha}\nabla_{\beta}R,
   respectively *)
i[m_,m_,n_,n_,a,b]:= -cst/2;
i[m_,n_,m_,n_,a,b]:= 0;
i[m_,n_,n_,m_,a,b]:= cst/2;
i[m_,m_,n_,a,n_,b]:= crt/3 - (5*cst)/8 -
  (3*ddr1)/4 + ddr2/4 - (5*rst)/12;
i[m_,n_,m_,a,n_,b]:= -crt/6 - (3*cst)/8 -
  (3*ddr1)/4 + ddr2/4 - (5*rst)/12;
i[m_,n_,n_,a,m_,b]:= -crt/6 - cst/8 -
  (3*ddr1)/4 + ddr2/4 - (5*rst)/12;
i[m_,m_,a,n_,n_,b]:= crt/3 - (5*cst)/8 +
  ddr1/4 + ddr2/4 - (5*rst)/12;
i[m_,n_,a,m_,n_,b]:= -crt/6 + (7*cst)/8 +
  ddr1/4 - ddr2/4 + (7*rst)/12;
i[m_,n_,a,n_,m_,b]:= -crt/6 + (5*cst)/8 +
  ddr1/4 - ddr2/4 + (7*rst)/12;
i[m_,a,m_,n_,n_,b]:= crt/12 - (3*cst)/8 +
  ddr1/4 + ddr2/4 - (5*rst)/12;
i[m_,a,n_,m_,n_,b]:= -crt/6 + (3*cst)/8 +
  ddr1/4 - ddr2/4 + (7*rst)/12;
i[m_,a,n_,n_,m_,b]:= crt/12 + cst/8 +
  ddr1/4 - ddr2/4 + rst/12;
i[a,m_,m_,n_,n_,b]:= -crt/6 - cst/8 +
  ddr1/4 + ddr2/4 + rst/12;
i[a,m_,n_,m_,n_,b]:= crt/12 + cst/8 +
  ddr1/4 - ddr2/4 + rst/12;
i[a,m_,n_,n_,m_,b]:= crt/12 + cst/8 +
  ddr1/4 - ddr2/4 + rst/12;

v10[6,6,ny]=v9[6,6,ny]//. {
  r[m_,m_,n_,n_]->0,
  r[m_,n_,m_,n_]->-sc,
  r[m_,n_,n_,m_]->sc};
Print["v10 finished!"]
Clear[v9];

(* rt stands for the Ricci tensor
   R_{\alpha\beta} *)
r[a,b]=rt;
r[x_,x_]:=sc;

(* ordering of indices in r *)
r[y_,x_]:=r[x,y]/; OrderedQ[{x,y}]
r[a,x_,y_,b]:=-r[a,x,b,y];
r[x_,a,b,y_]:=-r[a,x,b,y];
r[x_,a,y_,b]:=r[a,x,b,y];
r[b,y_,x_,a]:=-r[a,x,b,y];
r[b,y_,a,x_]:=r[a,x,b,y];
r[y_,b,x_,a]:=r[a,x,b,y];
r[y_,b,a,x_]:=-r[a,x,b,y];

r[x_,a,y_,z_]:=-r[a,x,y,z];
r[x_,b,y_,z_]:=-r[b,x,y,z]/; FreeQ[{x,y,z},a]
r[x_,y_,a,z_]:=r[a,z,x,y];
r[x_,y_,b,z_]:=r[b,z,x,y]/; FreeQ[{x,y,z},a]
r[x_,y_,z_,a]:=-r[a,z,x,y];
r[x_,y_,z_,b]:=-r[b,z,x,y]/; FreeQ[{x,y,z},a]

(* contraction of 2 indices of s[4] *)
v11[6,6,ny]=v10[6,6,ny]//. {
  s[m_,m_,x_,y_]->r[x,y]/3,
  s[m_,x_,m_,y_]->r[x,y]/3,
  s[m_,x_,y_,m_]->-2*r[x,y]/3,
  s[x_,m_,m_,y_]->-2*r[x,y]/3,
  s[x_,m_,y_,m_]->r[x,y]/3,
  s[x_,y_,m_,m_]->r[x,y]/3};
Print["v11 finished!"]
Clear[v10];

(* contraction of 2 indices of r *)
v12[6,6,ny]=v11[6,6,ny]//. {
  r[m_,m_,x_,y_]->0,
```

```
  r[m_,x_,m_,y_]->-r[x,y],
  r[m_,x_,y_,m_]->r[x,y],
  r[x_,m_,m_,y_]->r[x,y],
  r[x_,m_,y_,m_]->-r[x,y],
  r[x_,y_,m_,m_]->0};
Print["v12 finished!"]
Clear[v11];

(* contraction of two Ricci tensors *)
(* rs stands for R_{\mu\nu}^2 *)
v13[6,6,ny]=v12[6,6,ny]//.
  {r[x_,y_]^2->rs,r[a,x_]*r[b,x_]->rst};
Print["v13 finished!"]
Clear[v12];

(* contraction of a Ricci tensor and a
   curvature tensor *)
v14[6,6,ny]=v13[6,6,ny]//. {
  r[x_,y_]*r[a,b,x_,y_]->0,
  r[x_,y_]*r[a,b,y_,x_]->0,
  r[x_,y_]*r[a,x_,b,y_]->crt,
  r[x_,y_]*r[a,y_,b,x_]->crt};
Print["v14 finished!"]
Clear[v13];

(* ordering of indices in s *)
s[a,x_,b,y_]:=s[a,b,x,y]
s[x_,a,b,y_]:=s[a,x,y,b]
s[x_,a,y_,b]:=s[a,x,b,y]
s[x_,y_,a,b]:=s[a,x,b,y]
s[b,x_,y_,a]:=s[a,x,y,b]
s[b,y_,a,x_]:=s[a,b,x,y]
s[y_,b,x_,a]:=s[a,b,x,y]
s[x_,b,a,y_]:=s[a,x,y,b]

s[x_,a,y_,z_]:=s[a,x,z,y];
s[x_,b,y_,z_]:=s[b,x,z,y]/; FreeQ[{x,y,z},a]
s[x_,y_,a,z_]:=s[a,x,z,y];
s[x_,y_,b,z_]:=s[b,x,z,y]/; FreeQ[{x,y,z},a]
s[x_,y_,z_,a]:=s[a,y,z,x];
s[x_,y_,z_,b]:=s[b,y,z,x]/; FreeQ[{x,y,z},a]

(* contraction of a Ricci tensor and a s[4] *)
v15[6,6,ny]=v14[6,6,ny]//. {
  r[x_,y_]*s[a,b,x_,y_]->-crt/3,
  r[x_,y_]*s[a,b,y_,x_]->-crt/3,
  r[x_,y_]*s[a,x_,b,y_]->-crt/3,
  r[x_,y_]*s[a,y_,b,x_]->-crt/3,
  r[x_,y_]*s[a,x_,y_,b]->2*crt/3,
  r[x_,y_]*s[a,y_,x_,b]->2*crt/3};
Print["v15 finished!"]
Clear[v14];

(* contraction of two s[4]'s *)
v16[6,6,ny]=v15[6,6,ny]//. {
  s[a,m1_,m2_,m3_]*s[b,m1_,m2_,m3_]->cst/3,
  s[a,m1_,m2_,m3_]*s[b,m1_,m3_,m2_]->-cst/6,
  s[a,m1_,m2_,m3_]*s[b,m2_,m1_,m3_]->cst/3,
  s[a,m1_,m2_,m3_]*s[b,m2_,m3_,m1_]->-cst/6,
  s[a,m1_,m2_,m3_]*s[b,m3_,m1_,m2_]->-cst/6,
  s[a,m1_,m2_,m3_]*s[b,m3_,m2_,m1_]->-cst/6};
Print["v16 finished!"]
Clear[v15];

(* contraction of a s[4] and a curvature tensor *)
v17[6,6,ny]=v16[6,6,ny]//. {
  s[a,m1_,m2_,m3_]*r[b,m1_,m2_,m3_]->-cst/2,
  s[a,m1_,m2_,m3_]*r[b,m1_,m3_,m2_]->cst/2,
  s[a,m1_,m2_,m3_]*r[b,m2_,m1_,m3_]->-cst/2,
  s[a,m1_,m2_,m3_]*r[b,m2_,m3_,m1_]->cst/2,
s[a,m1_,m2_,m3_]*r[b,m3_,m1_,m2_]->0,
s[a,m1_,m2_,m3_]*r[b,m3_,m2_,m1_]->0};
Print["v17 finished!"]
Clear[v16];

(* ordering of dummy indices in s[4] *)
s[x_,m_,y_,z_]:=s[m,x,z,y]/;
  SameQ[m,Sort[{m,x,y,z}][[1]]] &&
  FreeQ[{m,x,y,z},a] && FreeQ[{m,x,y,z},b]
s[x_,y_,m_,z_]:=s[m,x,z,y]/;
  SameQ[m,Sort[{m,x,y,z}][[1]]] &&
  FreeQ[{m,x,y,z},a] && FreeQ[{m,x,y,z},b]
s[x_,y_,z_,m_]:=s[m,y,z,x]/;
  SameQ[m,Sort[{m,x,y,z}][[1]]] &&
  FreeQ[{m,x,y,z},a] && FreeQ[{m,x,y,z},b]
s[w_,y_,x_,z_]:=s[w,x,y,z]/;
  OrderedQ[{w,x,y}] &&
  FreeQ[{w,x,y,z},a] && FreeQ[{w,x,y,z},b]

(* contraction of two s[4]'s
   (with no uncontracted indices) *)
(* we denote R_{\mu\nu\rho\tau}
   R^{\mu\nu\rho\tau} as cs *)
v18[6,6,ny]=v17[6,6,ny]//. {
  s[m1_,m2_,m3_,m4_]^2->cs/3,
  s[m1_,m2_,m3_,m4_]*s[m1_,m2_,m4_,m3_]->-cs/6,
  s[m1_,m2_,m3_,m4_]*s[m1_,m3_,m2_,m4_]->cs/3,
  s[m1_,m2_,m3_,m4_]*s[m1_,m3_,m4_,m2_]->-cs/6,
  s[m1_,m2_,m3_,m4_]*s[m1_,m4_,m2_,m3_]->-cs/6,
  s[m1_,m2_,m3_,m4_]*s[m1_,m4_,m3_,m2_]->-cs/6,
  s[m1_,m2_,m4_,m3_]*s[m1_,m3_,m4_,m2_]->-cs/6};
Print["v18 finished!"]
Clear[v17];

(* contraction of a s[4] and a curvature tensor
   (with no uncontracted indices) *)
v19[6,6,ny]=v18[6,6,ny]//. {
  r[m1_,m2_,m3_,m4_]*s[m1_,m2_,m3_,m4_]->-cs/2,
  r[m1_,m2_,m3_,m4_]*s[m1_,m2_,m4_,m3_]->cs/2,
  r[m1_,m2_,m3_,m4_]*s[m1_,m3_,m2_,m4_]->-cs/2,
  r[m1_,m2_,m3_,m4_]*s[m1_,m3_,m4_,m2_]->0,
  r[m1_,m2_,m3_,m4_]*s[m1_,m4_,m2_,m3_]->cs/2,
  r[m1_,m2_,m3_,m4_]*s[m1_,m4_,m3_,m2_]->0};
Print["v19 finished!"]
Clear[v18];
```

```
(* contraction of s[6]
   (with no uncontracted indices) *)
(* ddr0 stands for \Box R *)
s[m_,m_,n_,n_,o_,o_]:=
  (4*cs)/15 + (2*ddr0)/5 + rs/15;
s[m_,n_,m_,n_,o_,o_]:=
  (-7*cs)/30 + (2*ddr0)/5 - (4*rs)/15;
s[m_,n_,n_,m_,o_,o_]:=
  (-11*cs)/15 + (2*ddr0)/5 - (4*rs)/15;
s[m_,m_,n_,o_,n_,o_]:=
  (4*cs)/15 + (2*ddr0)/5 + rs/15;
s[m_,n_,m_,o_,n_,o_]:=
  (-7*cs)/30 + (2*ddr0)/5 - (4*rs)/15;
s[m_,n_,n_,o_,m_,o_]:=
  (-11*cs)/15 + (2*ddr0)/5 - (4*rs)/15;
s[m_,m_,o_,n_,n_,o_]:=
  (4*cs)/15 - (3*ddr0)/5 + (2*rs)/5;
s[m_,n_,o_,m_,n_,o_]:=
  (-7*cs)/30 - ddr0/10 - (4*rs)/15;
s[m_,n_,o_,n_,m_,o_]:=
  (-7*cs)/30 - ddr0/10 - (4*rs)/15;
s[m_,o_,m_,n_,n_,o_]:=
  (4*cs)/15 - (3*ddr0)/5 + (2*rs)/5;
s[m_,o_,n_,m_,n_,o_]:=
  (4*cs)/15 - ddr0/10 + rs/15;
s[m_,o_,n_,n_,m_,o_]:=
  (4*cs)/15 - ddr0/10 + rs/15;
s[o_,m_,m_,n_,n_,o_]:=
  (4*cs)/15 - (3*ddr0)/5 - (4*rs)/15;
s[o_,m_,n_,m_,n_,o_]:=
  (4*cs)/15 - ddr0/10 + (2*rs)/5;
s[o_,m_,n_,n_,m_,o_]:=
  (4*cs)/15 - ddr0/10 + (2*rs)/5;

(* reordering of a and b *)
s[z1___,b,z2___,a,z3___]:=s[z1,a,z2,b,z3]

(* contraction of s[6] *)
s[m_,m_,n_,n_,a,b]:=
  (8*crt)/45 + (4*cst)/15 + (3*ddr1)/5 -
  ddr2/5 + (11*rst)/45;
s[m_,n_,m_,n_,a,b]:=
  (23*crt)/45 - (7*cst)/30 + (3*ddr1)/5 -
  ddr2/5 + (11*rst)/45;
s[m_,n_,n_,m_,a,b]:=
  (23*crt)/45 - (11*cst)/15 + (3*ddr1)/5 -
  ddr2/5 + (11*rst)/45;
s[m_,m_,n_,a,n_,b]:=
  (8*crt)/45 + (4*cst)/15 + (3*ddr1)/5 -
  ddr2/5 + (11*rst)/45;
s[m_,n_,m_,a,n_,b]:=
  (23*crt)/45 - (7*cst)/30 + (3*ddr1)/5 -
  ddr2/5 + (11*rst)/45;
s[m_,n_,n_,a,m_,b]:=
  (23*crt)/45 - (11*cst)/15 + (3*ddr1)/5 -
  ddr2/5 + (11*rst)/45;
s[m_,m_,a,n_,n_,b]:=
  (8*crt)/45 + (4*cst)/15 - (2*ddr1)/5 -
  ddr2/5 + (26*rst)/45;
s[m_,n_,a,m_,n_,b]:=
  (-22*crt)/45 - (7*cst)/30 - (2*ddr1)/5 +
  (3*ddr2)/10 - (34*rst)/45;
s[m_,n_,a,n_,m_,b]:=
  (-22*crt)/45 - (7*cst)/30 - (2*ddr1)/5 +
  (3*ddr2)/10 - (34*rst)/45;
s[m_,a,m_,n_,n_,b]:=
  (8*crt)/45 + (4*cst)/15 - (2*ddr1)/5 -
  ddr2/5 + (26*rst)/45;
s[m_,a,n_,m_,n_,b]:=
  (-22*crt)/45 + (4*cst)/15 - (2*ddr1)/5 +
  (3*ddr2)/10 - (19*rst)/45;
s[m_,a,n_,n_,m_,b]:=
  (-22*crt)/45 + (4*cst)/15 - (2*ddr1)/5 +
  (3*ddr2)/10 - (19*rst)/45;
s[a,m_,m_,n_,n_,b]:=
  (8*crt)/45 + (4*cst)/15 - (2*ddr1)/5 -
  ddr2/5 - (4*rst)/45;
s[a,m_,n_,m_,n_,b]:=
  (-22*crt)/45 + (4*cst)/15 - (2*ddr1)/5 +
  (3*ddr2)/10 - (4*rst)/45;
s[a,m_,n_,n_,m_,b]:=
  (-22*crt)/45 + (4*cst)/15 - (2*ddr1)/5 +
  (3*ddr2)/10 - (4*rst)/45;
s[m_,m_,n_,a,b,n_]:=
  (-37*crt)/45 + (4*cst)/15 - (9*ddr1)/10 +
  (3*ddr2)/10 - (19*rst)/45;
s[m_,n_,m_,a,b,n_]:=
  (-37*crt)/45 + (4*cst)/15 - (9*ddr1)/10 +
  (3*ddr2)/10 - (19*rst)/45;
s[m_,n_,n_,a,b,m_]:=
  (-7*crt)/45 + (4*cst)/15 - (9*ddr1)/10 +
  (3*ddr2)/10 - (19*rst)/45;
s[m_,m_,a,n_,b,n_]:=
  (-22*crt)/45 + (4*cst)/15 + ddr1/10 +
  (3*ddr2)/10 - (19*rst)/45;
s[m_,n_,a,m_,b,n_]:=
  (23*crt)/45 + (4*cst)/15 + ddr1/10 -
  ddr2/5 + (26*rst)/45;
s[m_,n_,a,n_,b,m_]:=
  (8*crt)/45 + (4*cst)/15 + ddr1/10 -
  ddr2/5 + (26*rst)/45;
s[m_,a,m_,n_,b,n_]:=
  (-7*crt)/45 - (7*cst)/30 + ddr1/10 +
  (3*ddr2)/10 - (19*rst)/45;
s[m_,a,n_,m_,b,n_]:=
  (38*crt)/45 - (7*cst)/30 + ddr1/10 -
  ddr2/5 + (26*rst)/45;
s[m_,a,n_,n_,b,m_]:=
  (-22*crt)/45 + (4*cst)/15 + ddr1/10 -
  ddr2/5 - (4*rst)/45;
s[a,m_,m_,n_,b,n_]:=
  (8*crt)/45 - (11*cst)/15 + ddr1/10 +
  (3*ddr2)/10 - (4*rst)/45;
s[a,m_,n_,m_,b,n_]:=
  (8*crt)/45 - (7*cst)/30 + ddr1/10 -
  ddr2/5 - (4*rst)/45;
```

```
s[a,m_,n_,n_,b,m_]:=
  (-7*crt)/45 + (4*cst)/15 + ddr1/10 -
  ddr2/5 - (4*rst)/45;
s[m_,m_,a,b,n_,n_]:=
  (-22*crt)/45 + (4*cst)/15 + ddr1/10 +
  (3*ddr2)/10 - (19*rst)/45;
s[m_,n_,a,b,m_,n_]:=
  (23*crt)/45 + (4*cst)/15 + ddr1/10 -
  ddr2/5 + (26*rst)/45;
s[m_,n_,a,b,n_,m_]:=
  (8*crt)/45 + (4*cst)/15 + ddr1/10 -
  ddr2/5 + (26*rst)/45;
s[m_,a,m_,b,n_,n_]:=
  (-7*crt)/45 - (7*cst)/30 + ddr1/10 +
  (3*ddr2)/10 - (19*rst)/45;
s[m_,a,n_,b,m_,n_]:=
  (38*crt)/45 - (7*cst)/30 + ddr1/10 -
  ddr2/5 + (26*rst)/45;
s[m_,a,n_,b,n_,m_]:=
  (-22*crt)/45 + (4*cst)/15 + ddr1/10 -
  ddr2/5 - (4*rst)/45;
s[a,m_,m_,b,n_,n_]:=
  (8*crt)/45 - (11*cst)/15 + ddr1/10 +
  (3*ddr2)/10 - (4*rst)/45;
s[a,m_,n_,b,m_,n_]:=
  (8*crt)/45 - (7*cst)/30 + ddr1/10 -
  ddr2/5 - (4*rst)/45;
s[a,m_,n_,b,n_,m_]:=
  (-7*crt)/45 + (4*cst)/15 + ddr1/10 -
  ddr2/5 - (4*rst)/45;
s[m_,a,b,m_,n_,n_]:=
  (-7*crt)/45 - (11*cst)/15 + ddr1/10 +
  (3*ddr2)/10 - (19*rst)/45;
s[m_,a,b,n_,m_,n_]:=
  (-7*crt)/45 - (11*cst)/15 + ddr1/10 +
  (3*ddr2)/10 - (19*rst)/45;
s[m_,a,b,n_,n_,m_]:=
  (23*crt)/45 + (4*cst)/15 + ddr1/10 -
  (7*ddr2)/10 + (11*rst)/45;
s[a,m_,b,m_,n_,n_]:=
  (8*crt)/45 - (7*cst)/30 + ddr1/10 +
  (3*ddr2)/10 - (4*rst)/45;
s[a,m_,b,n_,m_,n_]:=
  (8*crt)/45 - (7*cst)/30 + ddr1/10 +
  (3*ddr2)/10 - (4*rst)/45;
s[a,m_,b,n_,n_,m_]:=
  (-7*crt)/45 + (4*cst)/15 + ddr1/10 -
  (7*ddr2)/10 + (11*rst)/45;
s[a,b,m_,m_,n_,n_]:=
  (8*crt)/45 + (4*cst)/15 + ddr1/10 +
  (3*ddr2)/10 + (11*rst)/45;
s[a,b,m_,n_,m_,n_]:=
  (8*crt)/45 + (4*cst)/15 + ddr1/10 +
  (3*ddr2)/10 + (11*rst)/45;
s[a,b,m_,n_,n_,m_]:=
  (-7*crt)/45 + (4*cst)/15 + ddr1/10 -
  (7*ddr2)/10 + (11*rst)/45;

(* substitution into s1, ... , s8 *)
s1=(-8*crt)/15 - (14*cst)/5 + (6*ddr1)/5 +
  (18*ddr2)/5 - (7*cs*mt)/10 +
  (6*ddr0*mt)/5 - (7*mt*rs)/15 - (12*rst)/5;
s2=(32*crt)/15 - (4*cst)/5 - (4*ddr1)/5 -
  (12*ddr2)/5 - (cs*mt)/5 - (4*ddr0*mt)/5 -
  (2*mt*rs)/15 + (8*rst)/5;
s3=(-8*crt)/15 + (16*cst)/5 - (4*ddr1)/5 -
  (12*ddr2)/5 + (4*cs*mt)/5 - (4*ddr0*mt)/5 +
  (8*mt*rs)/15 + (8*rst)/5;
s4=(22*crt)/15 + (16*cst)/5 - (9*ddr1)/5 -
  (27*ddr2)/5 + (4*cs*mt)/5 - (9*ddr0*mt)/5 +
  (8*mt*rs)/15 + (18*rst)/5;
s5=(-38*crt)/15 + (16*cst)/5 + ddr1/5 +
  (3*ddr2)/5 + (4*cs*mt)/5 + (ddr0*mt)/5 +
  (8*mt*rs)/15 - (2*rst)/5;
s6=(2*crt)/15 - (4*cst)/5 + ddr1/5 +
  (3*ddr2)/5 - (cs*mt)/5 + (ddr0*mt)/5 -
  (2*mt*rs)/15 - (2*rst)/5;
s7=(22*crt)/15 - (14*cst)/5 + ddr1/5 +
  (3*ddr2)/5 - (7*cs*mt)/10 + (ddr0*mt)/5 -
  (7*mt*rs)/15 - (2*rst)/5;
s8=(14*crt)/5 - (24*cst)/5 + ddr1/5 +
  (3*ddr2)/5 - (6*cs*mt)/5 + (ddr0*mt)/5 -
  (4*mt*rs)/5 - (2*rst)/5;

(* substitution into t1,t2,t3 *)
t1=-4*crt/9+2*cst/3+cs*mt/6+mt*rs/9;
t2=8*crt/9-4*cst/3-cs*mt/3-2*mt*rs/9;
t3=-16*crt/9+8*cst/3+2*cs*mt/3+4*mt*rs/9;

(* substitution into ricci^2
   (g[l1,l2,l3,l4,a,b]r[l1,l2]r[l3,l4]) *)
temp=sub6//.
  ricci^2->mt*sc^2+2*mt*rs+4*rt*sc+8*rst;
Clear[sub6];

cont40[6,6,ny]=
  Expand[v19[6,6,ny]]+Expand[temp]

myfname0=
  StringJoin["cont406",ToString[ny],"_r.txt"]
Save[myfname0,cont40]
```

Appendix E. Recombination of terms

```
SetDirectory["D:\\trace_anomaly\\4dim"]
Get["param_r.txt"]

SetDirectory["cont023"]
Get["cont0230_r.txt"]
Get["cont0231_r.txt"]
Get["cont0232_r.txt"]
Get["cont0233_r.txt"]
r023=cont02[3,0,0]*p[3,0,0]+
cont02[3,0,1]*(p[3,0,0]-p[3,0,1])+
cont02[3,0,2]*
(p[3,0,0]-2*p[3,0,1]+p[3,0,2])+
```

```
cont02[3,0,3]*
(p[3,0,0]-3*p[3,0,1]+3*p[3,0,2]-p[3,0,3])
Clear[cont02]

ResetDirectory[]

 'r024,r025 are computed like as r023

r02=Factor[(r023+r024+r025)//. c->(a-1)/a]
Print["r02 finished!"]

 'r21,r40 are computed like as r02

r=Factor[r02+r21+r40]

(* coefficients of independent tensors *)
rmt=Factor[Coefficient[r,mt,1]]
rrt=Factor[Coefficient[r,rt,1]]
rcst=Factor[Coefficient[r,cst,1]]
rrst=Factor[Coefficient[r,rst,1]]
rcrt=Factor[Coefficient[r,crt,1]]
rd1=Factor[Coefficient[r,ddr1,1]]
rd2=Factor[Coefficient[r,ddr2,1]]

ResetDirectory[]
Save["recomb_r.txt",r02,r21,r40,r,rmt,rrt,
  rcst,rrst,rcrt,rd1,rd2]

(* coefficients of independent scalars *)
cmt[1]=Factor[rmt//. w->2+e]
cmt[2]=Factor[cmt[1]//. {a^e->1+e*Log[a],
  a^(e+k_)->a^k*(1+e*Log[a])}]
cmt[3]=cmt[2]//. e->0

 'like substitutions for rrt, rcst, rrst,
  rcrt, rd1, rd2

Save["recomb_r.txt",cmt,crt,ccst,crst,ccrt,
  cd1,cd2]
```

References and Note

[1] A.O. Barvinsky and G.A. Vilkovisky, *Phys. Rep.*, Vol.119, 1985, p.1.

[2] L.S. Brown and J.P. Cassidy, *Phys. Rev. D*, Vol.15, 1977, p.2810; J.S. Dowker and R. Chritchley, *Phys. Rev. D*, Vol.16, 1977, p.3390; S.M. Christensen and M.J. Duff, *Nucl. Phys. B*, Vol.154, 1979, p301.

[3] R. Endo, *Prog. Theor. Phys.*, Vol.71, 1984, p.1366.

[4] V.P. Gusynin and E.V. Gorbar, *Phys. Lett. B*, Vol.270, 1991, p.29.

[5] K. Seo, *Bull. Gifu City Wom. Col.*, No.50, 2001, p.63; *ibid*, No.51, 2002, p.89.

[6] S. Wolfram, *Mathematica: A System for Doing Mathematics by Computer* (Addison-Wesley), 1988.

[7] M.J. Booth, hep-th/9803113, 1998.

[8] I.G. Avramidi, *Nucl. Phys. B*, Vol.355, 1991, p.712, Erratum-ibid. Vol.509, 1998, p.557; I.G. Avramidi and R. Schimming, in *Leipzig 1995, Proceedings, Quantum field theory under the influence of external conditions*, 1995, p.150; I.G. Avramidi, hep-th/9510140, 1995.

[9] K. Fujikawa, *Phys. Rev. Lett.*, Vol.44, 1980, p.1733; *Phys. Rev. D* Vol.23, 1981, p.2262.

[10] K. Seo, *Bull. Gifu City Wom. Col.*, No.45, 1996, p.69.

[11] In Ref.5, the overall factors of the trace anomaly are miscounted. The factors should be $\frac{1}{4\pi}$ and $\frac{1}{16\pi^2}$ in two and four space-time dimensions, respectively, instead of $\frac{1}{2\pi}$ and $\frac{1}{4\pi^2}$.

[12] N.D. Birrell and P.C.W. Davies, *Quantum fields in curved space* (Cambridge University Press), 1982.

（提出期日　2003 年 3 月 5 日）