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Evaluation of Trace Anomalies in 6 Space-Time Dimensions by Mathematica

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Abstract

Trace anomalies for a scalar and a Dirac field theoris in 6 space-time dimensions are evaluated by making use of our result for the third heat kernel coefficient presented in the previous paper. Among 17 possible terms, 7 terms are eliminated by adding local counter terms to the original action. The results for a scalar thoery are compared with the results by other people.

Keywords:heat kernel, trace anomaly, 6 dimensions, Mathematica

I Introduction

In the prvious paper[1], we present a evaluation method of the third heat kernel coefficient in a Dirac field theory. The trace anomaly of the energy-momentum tensor in 6 space-time dimensions is derivable from the coefficient as follows:

$$\sqrt{-g}\, g^{\mu\nu} < T_{\mu\nu}(x) > \\ = -i \lim_{x'\to x} \mathrm{tr}\big[\, e^{-\epsilon Y^{(1/2)}} I(x,x')\delta^{(n)}(x,x')\big]\,, \qquad (1.1)$$

with

$$Y^{(1/2)} = \not{D}^2 = g^{\mu\nu}\, D_\mu\, D_\nu + \xi\, R\,, \qquad (1.2)$$

where $\xi = \frac{1}{4}$. If we retain ξ as a free parameter, the trace anomaly for the scalar theory,

$$\sqrt{-g}\, g^{\mu\nu} < T_{\mu\nu}(x) > \\ = i \lim_{x'\to x} \mathrm{tr}\big[\, e^{-\epsilon Y^{(0)}} \delta^{(n)}(x,x')\big]\,, \qquad (1.3)$$

with

$$Y^{(0)} = g^{\mu\nu}\, \nabla_\mu\, \nabla_\nu + \xi\, R = \Box + \xi\, R\,, \qquad (1.4)$$

is known by neglecting the derivative terms of I and setting the parameter ξ as $(n-2)/[4(n-1)]$.

The third heat kernel coefficient was evaluated by Gilkey[2] at first. There are some discrepancy between our result and Gilkey's one. As for the coefficients d_8 and d_{11} in our notation of Ref.1, some miss typings of our Mathematica program were found, and the discrepancy vanished[3]. As for the coefficients c_{13} and c_{14}, our result is consistent with the results by Avramidi[4] and Barvinsky *et. al.*[5] They are giving only the integrated expression instead of the local one, and there exists some ambiguity on the value of the coefficients. In spite of their claim that their results are consistent with Gilkey's one, their results are rather consitent with our's.

The trace anomaly in 4 space-time dimensions, there exist 4 possible terms. Among them the term proportional to $\Box R$ can be eliminated by adding a local counter term. In 6 space-time dimensions, there exist 17 possible terms. First we must know whether these terms can be eliminated by couter terms or not. The contributuion to the trace anomaly of the possible local counter terms are evaluated with the help of Mathematica.

Next we select terms to be left and determine their coefficients. Finally we compare our results for a scalar theory with those by other people [6,7,8].

II Elimination of spurious anomalies by local counter terms

Under the local Weyl transformation $\delta g^{\mu\nu} = 2\alpha g^{\mu\nu}$, the curvature tensor and its covariant derivative transform as follows:

$$\int d^n x\, F^{\mu\nu\rho\sigma}\, \delta R_{\mu\nu\rho\sigma} \\ = \int d^n x\, \alpha \big\{ -2R_{\mu\nu\rho\sigma} + g_{\mu\rho}\nabla_\sigma\nabla_\nu - g_{\nu\rho}\nabla_\sigma\nabla_\mu \\ -g_{\mu\sigma}\nabla_\rho\nabla_\nu + g_{\nu\sigma}\nabla_\rho\nabla_\mu \big\} F^{\mu\nu\rho\sigma}\,, \qquad (2.1)$$

$$\int d^n x\, F^{\mu\nu\rho\sigma\tau}\, \delta(\nabla_\mu R_{\nu\rho\sigma\tau}) \\ = \int d^n x\, \alpha \Big[\big\{ 2R_{\nu\rho\sigma\tau} - g_{\nu\sigma}\nabla_\tau\nabla_\rho + g_{\rho\sigma}\nabla_\tau\nabla_\nu \\ +g_{\nu\tau}\nabla_\sigma\nabla_\rho - g_{\rho\tau}\nabla_\sigma\nabla_\nu \big\} \nabla_\mu F^{\mu\nu\rho\sigma\tau} \\ -4\nabla_\mu(R_{\nu\rho\sigma\tau}F^{\mu\nu\rho\sigma\tau}) - \nabla_\nu(R_{\mu\rho\sigma\tau}F^{\mu\nu\rho\sigma\tau}) \\ -\nabla_\rho(R_{\nu\mu\sigma\tau}F^{\mu\nu\rho\sigma\tau}) - \nabla_\sigma(R_{\nu\rho\mu\tau}F^{\mu\nu\rho\sigma\tau}) \\ -\nabla_\tau(R_{\nu\rho\sigma\mu}F^{\mu\nu\rho\sigma\tau}) + g_{\mu\nu}\nabla^\epsilon(R_{\epsilon\rho\sigma\tau}F^{\mu\nu\rho\sigma\tau}) \\ +g_{\mu\rho}\nabla^\epsilon(R_{\nu\epsilon\sigma\tau}F^{\mu\nu\rho\sigma\tau}) + g_{\mu\sigma}\nabla^\epsilon(R_{\nu\rho\epsilon\tau}F^{\mu\nu\rho\sigma\tau}) \\ +g_{\mu\tau}\nabla^\epsilon(R_{\nu\rho\sigma\epsilon}F^{\mu\nu\rho\sigma\tau}) \Big]\,, \qquad (2.2)$$

where F's are arbitrary tensors. These formulas are coded by Mathematica as shown in Appendix A, and the contributions of local counter terms to the trace anomaly are obtained in the following. For simplicity, we use the following notation,

$$\Delta\mathcal{L} \equiv \frac{1}{\sqrt{-g}} g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}} \int d^n x \sqrt{-g}\, \mathcal{L}\,, \qquad (2.3)$$

i	$\mathcal{L}_i$ (symbol)	c_{i1}	c_{i2}	c_{i3}	c_{i4}	c_{i5}	c_{i6}	c_{i7}
1	R^3 (sc^3)	0	$6(1-n)$	0	0	0	0	0
2	$R\,R^2_{\mu\nu}$ (sc*rsq)	0	$-(1+\frac{n}{2})$	$2-n$	$2(1-n)$	0	0	0
3	$R\,R^2_{\mu\nu\rho\sigma}$ (sc*csq)	0	-2	-4	0	$2(1-n)$	0	0
4	$R^\mu{}_\nu\,R^\nu{}_\rho\,R^\rho{}_\mu$ (rcu)	0	0	$\frac{3}{2}(1-\frac{n}{2})$	-3	0	$3(1-\frac{n}{2})$	0
5	$R_{\mu\rho}\,R_{\nu\sigma}\,R^{\mu\nu\rho\sigma}$ (crsq)	0	$\frac{1}{2}$	$\frac{1}{2}$	n	0	$1-n$	$n-2$
6	$R^\mu{}_\nu\,R_{\mu\rho\sigma\tau}\,R^{\nu\rho\sigma\tau}$ (rcsq)	0	0	-1	-2	$-\frac{n+2}{4}$	0	$-n$
7	$R_{\mu\nu\rho\sigma}\,R^{\mu\nu}{}_{\tau\lambda}\,R^{\rho\sigma\tau\lambda}$ (ccu1)	0	0	0	0	3	0	12
8	$R_{\mu\rho\nu\sigma}\,R^\mu{}_\tau{}^\nu{}_\lambda\,R^{\rho\tau\sigma\lambda}$ (ccu2)	0	0	0	3	$-\frac{3}{4}$	-3	0
9	$(\nabla_\mu R)^2$ (dscsq)	$2(n-1)$	-2	0	0	0	0	0
10	$(\nabla_\mu R_{\nu\rho})^2$ (drsq2)	$\frac{n}{2}$	0	$3-n$	$n-4$	0	$2(1-n)$	$2(n-2)$
11	$(\nabla_\mu R_{\nu\rho\sigma\tau})^2$ (dcsq)	2	0	-4	4	-4	-8	16

Table 1: The coefficients c's in Eq.(2.14) which describe the contributions of the local counter terms to the trace anomaly.

where $\mathcal{L}$ is an arbitrary scalar density. In 4 space-time dimensions, there exist 3 possible counter terms and their conformal variations take the following expressions:

$$\Delta R^2 = 2(1-n)\Box R + (2-\frac{n}{2})R^2\,, \qquad (2.4)$$

$$\Delta (R_{\alpha\beta})^2 = -\frac{n}{2}\Box R + (2-\frac{n}{2})(R_{\alpha\beta})^2\,, \qquad (2.5)$$

$$\Delta (R_{\alpha\beta\gamma\delta})^2 = -2\Box R + (2-\frac{n}{2})(R_{\alpha\beta\gamma\delta})^2\,. \qquad (2.6)$$

Therefore the trace anomaly term with the tensor structure $\Box R$ can be eliminated by the local counter term proportional to R^2. In 6 space-time dimensions, there are 11 possible local counter terms and their conformal variations can eliminate the following 7 tensor structures,

$$T_1 \equiv \Box^2 R\,, \qquad (2.7)$$

$$T_2 \equiv \nabla^\mu(R\nabla_\mu R) = R\Box R + (\nabla_\mu R)^2\,, \qquad (2.8)$$

$$T_3 \equiv \nabla_\mu(R^{\mu\nu}\nabla_\nu R) = R^{\mu\nu}\nabla_\mu\nabla_\nu R + \frac{1}{2}(\nabla_\mu R)^2\,, \qquad (2.9)$$

$$T_4 \equiv \nabla^\mu(R^{\nu\rho}\nabla_\mu R_{\nu\rho}) = R^{\mu\nu}\Box R_{\mu\nu} + (\nabla_\mu R_{\nu\rho})^2\,, \qquad (2.10)$$

$$\begin{aligned} T_5 &\equiv \nabla^\mu(R^{\nu\rho\sigma\tau}\nabla_\mu R_{\nu\rho\sigma\tau}) \\ &= R^{\mu\nu\rho\sigma}\Box R_{\mu\nu\rho\sigma} + (\nabla_\mu R_{\nu\rho\sigma\tau})^2\,, \end{aligned} \qquad (2.11)$$

$$\begin{aligned} T_6 &\equiv \nabla^\mu(R^{\nu\rho}\nabla_\nu R_{\mu\rho}) \\ &= (\nabla^\mu R^{\nu\rho})\nabla_\nu R_{\mu\rho} + R^{\nu\rho}\nabla^\mu\nabla_\nu R_{\mu\rho} \\ &= (\nabla^\mu R^{\nu\rho})\nabla_\nu R_{\mu\rho} + \frac{1}{2}R^{\mu\nu}\nabla_\mu\nabla_\nu R - R^\mu{}_\nu R^\nu{}_\rho R^\rho{}_\mu \\ &\quad -R_{\mu\nu}R_{\rho\sigma}R^{\mu\rho\nu\sigma}\,, \end{aligned} \qquad (2.12)$$

$$\begin{aligned} T_7 &\equiv \nabla_\mu(R^{\mu\rho\sigma\nu}\nabla_\nu R_{\rho\sigma}) \\ &= \nabla_\mu R^{\mu\rho\sigma\nu}\nabla_\nu R_{\rho\sigma} + R^{\mu\rho\sigma\nu}\nabla_\mu\nabla_\nu R_{\rho\sigma} \\ &= (\nabla_\mu R_{\nu\rho})^2 - (\nabla^\mu R^{\nu\rho})\nabla_\nu R_{\mu\rho} + \frac{1}{4}R^{\mu\nu\rho\sigma}\Box R_{\mu\nu\rho\sigma} \\ &\quad +\frac{1}{2}R^\mu{}_\nu R_{\mu\rho\sigma\tau}R^{\rho\sigma\tau} + \frac{1}{4}R_{\mu\nu\rho\sigma}R^{\mu\nu}{}_{\tau\lambda}R^{\rho\sigma\tau\lambda} \\ &\quad +R_{\mu\rho\nu\sigma}R^\mu{}_\tau{}^\nu{}_\lambda R^{\rho\tau\sigma\lambda}\,. \end{aligned} \qquad (2.13)$$

The contributions of 11 local counter terms to the trace anomaly are summarized as

$$\Delta\mathcal{L}_i = \sum_{j=1}^{7} c_{ij} T_j + (3 - \frac{n}{2})\mathcal{L}_i \,.(i = 1, 2, \cdots 11) \quad (2.14)$$

The coefficients c_{ij}'s are shown in Table 1. Among 11 terms, 7 terms are sufficient to eliminate spurious anomaly terms. For example, $i = 1, 2, 3, 4, 7, 8, 9$.

III Results and discussions

In the previous section, it is understood that 7 terms can be eliminated by addition of local counter terms. Now we select 10 terms to be left, and we define the 10 coefficients as follows:

$$\begin{aligned}
&< T^{\mu}{}_{\mu}(x) > \\
&= \frac{1}{64\pi^3}\{c_1\,(\nabla_\mu R)^2 + c_2\,(\nabla_\mu R_{\nu\rho\sigma\tau})^2 + c_3\,R^3 \\
&\quad + c_4\,R\,R^2_{\mu\nu} + c_5\,R\,R^2_{\mu\nu\rho\sigma} + c_6\,R^{\mu}{}_{\nu}\,R^{\nu}{}_{\rho}\,R^{\rho}{}_{\mu} \\
&\quad + c_7\,R_{\mu\rho}\,R_{\nu\sigma}\,R^{\mu\nu\rho\sigma} + c_8\,R^{\mu}{}_{\nu}\,R_{\mu\rho\sigma\tau}\,R^{\nu\rho\sigma\tau} \\
&\quad + c_9\,R_{\mu\nu\rho\sigma}\,R^{\mu\nu}{}_{\tau\lambda}\,R^{\rho\sigma\tau\lambda} \\
&\quad + c_{10}\,R_{\mu\rho\nu\sigma}\,R^{\mu}{}_{\tau}{}^{\nu}{}_{\lambda}\,R^{\rho\tau\sigma\lambda}\}\,. \qquad (3.1)
\end{aligned}$$

If we denote the coefficients before addition of local counter terms as

$$\begin{aligned}
&< T^{\mu}{}_{\mu}(x) >_{\rm bare} \\
&= \frac{1}{64\pi^3}\{a_1\,(\nabla_\mu R)^2 + a_2\,(\nabla_\mu R_{\nu\rho\sigma\tau})^2 + a_3\,R^3 \\
&\quad + a_4\,R\,R^2_{\mu\nu} + a_5\,R\,R^2_{\mu\nu\rho\sigma} + a_6\,R^{\mu}{}_{\nu}\,R^{\nu}{}_{\rho}\,R^{\rho}{}_{\mu} \\
&\quad + a_7\,R_{\mu\rho}\,R_{\nu\sigma}\,R^{\mu\nu\rho\sigma} + a_8\,R^{\mu}{}_{\nu}\,R_{\mu\rho\sigma\tau}\,R^{\nu\rho\sigma\tau} \\
&\quad + a_9\,R_{\mu\nu\rho\sigma}\,R^{\mu\nu}{}_{\tau\lambda}\,R^{\rho\sigma\tau\lambda} \\
&\quad + a_{10}\,R_{\mu\rho\nu\sigma}\,R^{\mu}{}_{\tau}{}^{\nu}{}_{\lambda}\,R^{\rho\tau\sigma\lambda} \\
&\quad + b_1\,\Box^2 R + b_2\,R\,\Box\,R + b_3\,R^{\mu\nu}\nabla_\mu\nabla_\nu R \\
&\quad + b_4\,R^{\mu\nu}\,\Box\,R_{\mu\nu} + b_5\,R^{\mu\nu\rho\sigma}\,\Box\,R_{\mu\nu\rho\sigma} \\
&\quad + b_6\,(\nabla_\mu R_{\nu\rho})^2 + b_7\,(\nabla^\mu R^{\nu\rho})\nabla_\nu R_{\mu\rho}\}\,, \qquad (3.2)
\end{aligned}$$

then c's are related with a's and b's as follows:

$$c_1 = a_1 - b_2 - \frac{1}{2}b_3 + \frac{1}{4}(-b_4 + b_6 + b_7)\,, \qquad (3.3)$$

$$c_2 = a_2 - b_5 + \frac{1}{4}(-b_4 + b_6)\,, \qquad (3.4)$$

$$c_i = a_i\,, \qquad (i = 3, 4, 5) \qquad (3.5)$$

$$c_6 = a_6 - b_4 + b_6 + b_7\,, \qquad (3.6)$$

$$c_7 = a_7 - b_4 + b_6 + b_7\,, \qquad (3.7)$$

$$c_8 = a_8 + \frac{1}{2}(b_4 - b_6)\,, \qquad (3.8)$$

$$c_9 = a_9 + \frac{1}{4}(b_4 - b_6)\,, \qquad (3.9)$$

$$c_{10} = a_{10} + b_4 - b_6\,. \qquad (3.10)$$

<table>
<tr><th rowspan="2">i</th><th rowspan="2">tensor
(symbol)</th><th colspan="2">c's value $(\times(-8)^f)$</th></tr>
<tr><th>scalar
$(f{=}0,\ \xi{=}\frac{1}{5})$</th><th>Dirac
$(f{=}1,\ \xi{=}\frac{1}{4})$</th></tr>
<tr><td rowspan="2">1</td><td rowspan="2">$(\nabla_\mu R)^2$
(dscsq)</td><td colspan="2">$\frac{11}{3360} - \frac{\xi}{30} + \frac{\xi^2}{12}$</td></tr>
<tr><td>$-\frac{1}{16800}$</td><td>$-\frac{1}{840}$</td></tr>
<tr><td rowspan="2">2</td><td rowspan="2">$(\nabla_\mu R_{\nu\rho\sigma\tau})^2$
(dcsq)</td><td colspan="2">$\frac{1}{3360} - \frac{f}{960}$</td></tr>
<tr><td>$\frac{1}{3360}$</td><td>$\frac{1}{168}$</td></tr>
<tr><td rowspan="2">3</td><td rowspan="2">R^3
(sc^3)</td><td colspan="2">$\frac{1}{1296} - \frac{\xi}{72} + \frac{\xi^2}{12} - \frac{\xi^3}{6}$</td></tr>
<tr><td>$-\frac{1}{162000}$</td><td>$\frac{1}{1296}$</td></tr>
<tr><td rowspan="2">4</td><td rowspan="2">$R\,R^2_{\mu\nu}$
(sc*rsq)</td><td colspan="2">$-\frac{1}{1080} + \frac{\xi}{180}$</td></tr>
<tr><td>$\frac{1}{5400}$</td><td>$-\frac{1}{270}$</td></tr>
<tr><td rowspan="2">5</td><td rowspan="2">$R\,R^2_{\mu\nu\rho\sigma}$
(sc*csq)</td><td colspan="2">$\frac{1}{1080} - \frac{\xi}{180} - \frac{f}{576} + \frac{f\cdot\xi}{96}$</td></tr>
<tr><td>$-\frac{1}{5400}$</td><td>$-\frac{7}{2160}$</td></tr>
<tr><td rowspan="2">6</td><td rowspan="2">$R^{\mu}{}_{\nu}\,R^{\nu}{}_{\rho}\,R^{\rho}{}_{\mu}$
(rcu)</td><td colspan="2">$-\frac{1}{4536}$</td></tr>
<tr><td>$-\frac{1}{4536}$</td><td>$\frac{1}{567}$</td></tr>
<tr><td rowspan="2">7</td><td rowspan="2">$R_{\mu\rho}\,R_{\nu\sigma}\,R^{\mu\nu\rho\sigma}$
(crsq)</td><td colspan="2">$\frac{1}{7560}$</td></tr>
<tr><td>$\frac{1}{7560}$</td><td>$-\frac{1}{945}$</td></tr>
<tr><td rowspan="2">8</td><td rowspan="2">$R^{\mu}{}_{\nu}\,R_{\mu\rho\sigma\tau}\,R^{\nu\rho\sigma\tau}$
(rcsq)</td><td colspan="2">$-\frac{1}{2160} + \frac{f}{1440}$</td></tr>
<tr><td>$-\frac{1}{2160}$</td><td>$-\frac{1}{540}$</td></tr>
<tr><td rowspan="2">9</td><td rowspan="2">$R_{\mu\nu\rho\sigma}\,R^{\mu\nu}{}_{\tau\lambda}\,R^{\rho\sigma\tau\lambda}$
(ccu1)</td><td colspan="2">$-\frac{61}{90720} + \frac{f}{576}$</td></tr>
<tr><td>$-\frac{61}{90720}$</td><td>$-\frac{193}{22680}$</td></tr>
<tr><td rowspan="2">10</td><td rowspan="2">$R_{\mu\rho\nu\sigma}\,R^{\mu}{}_{\tau}{}^{\nu}{}_{\lambda}\,R^{\rho\tau\sigma\lambda}$
(ccu2)</td><td colspan="2">$-\frac{13}{22680} + \frac{f}{360}$</td></tr>
<tr><td>$-\frac{13}{22680}$</td><td>$-\frac{10}{567}$</td></tr>
</table>

Table 2: The coefficients c's in Eq.(3.1) which describe the minimal trace anomaly. f is a parameter to discriminate the scalar case and the Dirac case, and taking the value 0 and 1, respectively. The factor (-8) for the Dirac case comes from the trace of the Dirac matrix and the fermion loop.

The formulas for the trace of the Dirac matrix are listed in Appendix B. Our results for the coefficients c's are shown in Table 2.

Bastianelli and Dass[6] have presented a simple method to evaluate the trace anomaly for a scalar theory. Their calculation is based on the paper by Bastianelli, Cuoghi and Nocetti[7], where the consistency condition for the trace anomaly is argued and it is concluded that only 4 coefficients are independent:

$$< T^{\mu}{}_{\mu} >= \frac{1}{64\pi^3}(aE_6 + c_1 I_1 + c_2 I_2 + c_3 I_3), \quad (3.11)$$

where E_6 is the topological Euler density, and I's are three independent Weyl invariants. In Ref.6, the coefficients have been determined by reducing the problem to a quantum mechanical one. On the other hand, Ichinose and Ikeda [8] have presented an algorithm to obtain the trace anomaly in higher dimensions, and have carried out the program in a 6 dimensional scalar thoery.

In order to comapare these results with our's, we rewrite E_6 and 3 Weyl invariants by 17 terms in Eq.(3.2). In fact, 3 terms are absent, $\Box^2 R, (\nabla_\mu R)^2$, and $R^{\mu\nu}\nabla_\mu\nabla_\nu R$. We determine 3 coefficients of the contributions of local counter terms to the trace anomaly, $\frac{1}{64\pi^3}\sum_{i=1}^{7}\alpha_i T_i$, to eliminate these 3 terms. Other 4 coefficients $\alpha_i (i=3,4,5,7)$ are kept as free parameters. Our results for a scalar theory are shown in Table 3, and are compared with those by Bastianelli and Dass[6], and with those by Ichinose and Ikeda[8]. Even if we take special values for the parameters $\alpha' s$, these results cannot be consistent with each other.

Hatzinikitas and Portugal[9] have also given the integrated trace anomaly, by carring out a supersymmetric quantum mechanical computation. They, however, set $\xi = \frac{2}{15}$, and their results cannot be directly compared with those by other people ($\xi = \frac{1}{5}$).

tensor (symbol)	our results	B.D.	I.I.
$R\,\Box R$ (sc*ddsc)	$\frac{1}{25200} - \frac{\alpha_3}{2}$	$\frac{1}{4200}$	$-\frac{11}{18900}$
$R^{\mu\nu}\,\Box R_{\mu\nu}$ (rddr1)	$\frac{1}{630} + \alpha_4$	$-\frac{1}{420}$	$\frac{11}{1890}$
$R^{\mu\nu\rho\sigma}\,\Box R_{\mu\nu\rho\sigma}$ (cddc)	$-\frac{1}{420} + \alpha_5 + \frac{\alpha_7}{4}$	$\frac{1}{420}$	$-\frac{11}{1890}$
$(\nabla_\mu R_{\nu\rho})^2$ (drsq2)	$\frac{1}{2520} + \alpha_4 + \alpha_7$	$\frac{1}{840}$	0
$(\nabla^\mu R^{\nu\rho})\,\nabla_\nu R_{\mu\rho}$ (drsq3)	$\frac{1}{900} - 2\alpha_3 - \alpha_7$	$-\frac{1}{420}$	0
$(\nabla_\mu R_{\nu\rho\sigma\tau})^2$ (dcsq)	$-\frac{1}{560} + \alpha_5$	$\frac{1}{840}$	0
R^3 (sc^3)	$-\frac{1}{16200}$	$-\frac{1}{16200}$	$\frac{47}{283500}$
$R\,R^2_{\mu\nu}$ (sc*rsq)	$\frac{1}{5400}$	$\frac{1}{5400}$	$\frac{17}{18900}$
$R\,R^2_{\mu\nu\rho\sigma}$ (sc*csq)	$-\frac{1}{5400}$	$-\frac{1}{5400}$	$-\frac{11}{2700}$
$R^\mu{}_\nu\,R^\nu{}_\rho\,R^\rho{}_\mu$ (rcu)	$-\frac{2}{14175} + 2\alpha_3$	$\frac{11}{11340}$	$\frac{1}{5670}$
$R_{\mu\rho}\,R_{\nu\sigma}\,R^{\mu\nu\rho\sigma}$ (crsq)	$\frac{1}{4725} + 2\alpha_3$	$-\frac{1}{756}$	$\frac{1}{315}$
$R^\mu{}_\nu\,R_{\mu\rho\sigma\tau}\,R^{\nu\rho\sigma\tau}$ (rcsq)	$-\frac{1}{945} + \frac{\alpha_7}{2}$	$-\frac{17}{7560}$	$\frac{1}{126}$
$R_{\mu\nu\rho\sigma}\,R^{\mu\nu}{}_{\tau\lambda}\,R^{\rho\sigma\tau\lambda}$ (ccu1)	$-\frac{11}{11340} + \frac{\alpha_7}{4}$	$\frac{71}{45360}$	$-\frac{11}{11340}$
$R_{\mu\rho\nu\sigma}\,R^\mu{}_\tau{}^\nu{}_\lambda\,R^{\rho\tau\sigma\lambda}$ (ccu2)	$-\frac{1}{567} + \alpha_7$	$\frac{47}{11340}$	$-\frac{1}{567}$

Table 3: Comparison of the coefficients of the trace anomaly for a scalar theory. B.D. and I.I. are the results of Ref.6 and Ref.8, respectively.

Appendix A. Variation of counter terms under the Weyl transformation

```
SetAttributes[P,Flat];
SetAttributes[g,Orderless];

(* conformal transformation, s is a fixed
 index. *)
var[P[x__]]:= Sum[Apply[P,
 Join[{var[{x}[[j]]]}, Delete[{x},j]]],
 {j,1,Length[{x}]}];
(* variation of the square root of the
 metric tensor
var[e]= -N*P[e];
(* variation of the metric tensor with upper
 indices *)
var[g[m_,n_]]:= 2*P[g[m,n]];
(* variation of the curvature tensor *)
var[r[m_,n_,o_,p_]]:= -2*P[r[m,n,o,p]] +
 P[g[m,o],d[p],d[n]] - P[g[m,p],d[o],d[n]] -
 P[g[n,o],d[p],d[m]] + P[g[n,p],d[o],d[m]];
(* variation of the derivative of the
 curvature tensor *)
var[r[m_,n_,o_,p_,q_]]:=
 2*P[r[n,o,p,q],d[m]] - P[g[n,p],d[q],
  d[o],d[m]] + P[g[n,q],d[p],d[o],d[m]] +
 P[g[o,p],d[q],d[n],d[m]] - P[g[o,q],d[p],
  d[n],d[m]] - 4*P[d[m],r[n,o,p,q]] -
 P[d[n],r[m,o,p,q]] - P[d[o],r[n,m,p,q]] -
 P[d[p],r[n,o,m,q]] - P[d[q],r[n,o,p,m]] +
 P[g[m,n],d[s],r[s,o,p,q]] + P[g[m,o],d[s],
  r[n,s,p,q]] + P[g[m,p],d[s],r[n,o,s,q]] +
 P[g[m,q],d[s],r[n,o,p,s]];
```

```
(* distribution rule and multiplication of
 constant *)
P[x___,y1_+y2_,z___]:= P[x,y1,z] + P[x,y2,z];
P[x___,c_*P[y__],z___]:= c*P[x,y,z];

counter=1
(* R^2 *)
f[1,counter]= var[P[e,g[m,n],g[o,p],
 r[m,o,p,n],g[a,b],g[c,d],r[a,c,d,b]]];
(* (R_{\mu\mu})^2 *)
f[2,counter]= var[P[e,g[m1,m2],g[n1,n2],
 g[o1,p1],g[o2,p2],r[m1,o1,p1,n1],
 r[m2,o2,p2,n2]]];
(* (R_{\mu\nu\rho\sigma})^2 *)
f[3,counter]= var[P[e,g[m1,m2],g[n1,n2],
 g[o1,o2],g[p1,p2],r[m1,n1,o1,p1],
```

```
 r[m2,n2,o2,p2]]];
(* R^3 *)
f[4,counter]= var[P[e,g[m1,n1],g[o1,p1],
 r[m1,o1,p1,n1],g[m2,n2],g[o2,p2],
 r[m2,o2,p2,n2],g[m3,n3],g[o3,p3],
 r[m3,o3,p3,n3]]];
(* R(R_{\mu\mu})^2 *)
f[5,counter]= var[P[e,g[m1,m2],g[n1,n2],
 g[o1,p1],g[o2,p2],g[m3,n3],g[o3,p3],
 r[m1,o1,p1,n1],r[m2,o2,p2,n2],
 r[m3,o3,p3,n3]]];
(* R(R_{\mu\nu\rho\sigma})^2 *)
f[6,counter]= var[P[e,g[m1,m2],g[n1,n2],
 g[o1,o2],g[p1,p2],g[m3,n3],g[o3,p3],
 r[m1,o1,p1,n1],r[m2,o2,p2,n2],
 r[m3,o3,p3,n3]]];
(* R_{\mu\nu}R_{\nu\rho}R_{\rho\mu} *)
f[7,counter]= var[P[e,g[m1,n3],g[n1,m2],
 g[n2,m3],g[o1,p1],g[o2,p2],g[o3,p3],
 r[m1,o1,p1,n1],r[m2,o2,p2,n2],
 r[m3,o3,p3,n3]]];
(* R_{\mu\nu}R_{\rho\sigma}
 R_{\mu\rho\nu\sigma} *)
f[8,counter]= var[P[e,g[m1,m3],g[n1,p3],
 g[m2,o3],g[n2,n3],g[o1,p1],g[o2,p2],
 r[m1,o1,p1,n1],r[m2,o2,p2,n2],
 r[m3,o3,p3,n3]]];
(* R_{\mu\nu}R_{\mu\alpha\beta\gamma}
 R_{\nu\alpha\beta\gamma} *)
f[9,counter]= var[P[e,g[m1,m2],g[n1,m3],
 g[o1,p1],g[n2,n3],g[o2,o3],g[p2,p3],
 r[m1,o1,p1,n1],r[m2,o2,p2,n2],
 r[m3,o3,p3,n3]]];
(* R_{\mu\nu\rho\sigma}R_{\mu\nu\alpha\beta}
 R_{\rho\sigma\alpha\beta} *)
f[10,counter]= var[P[e,g[m1,m2],g[o1,o2],
 g[p1,m3],g[n1,o3],g[n2,n3],g[p2,p3],
 r[m1,o1,p1,n1],r[m2,o2,p2,n2],
 r[m3,o3,p3,n3]]];
(* R_{\mu\rho\nu\sigma}R_{\mu\alpha\nu\beta}
 R_{\rho\alpha\sigma\beta} *)
f[11,counter]= var[P[e,g[m1,m2],g[o1,m3],
 g[p1,p2],g[n1,p3],g[n2,n3],g[o2,o3],
 r[m1,o1,p1,n1],r[m2,o2,p2,n2],
 r[m3,o3,p3,n3]]];
(* (\nabla_\mu R)^2 *)
f[12,counter]= var[P[e,g[x,y],g[m,n],g[o,p],
 r[x,m,o,p,n],g[a,b],g[c,d],r[y,a,c,d,b]]];
(* (\nabla_\mu R_{\nu\rho})^2 *)
f[13,counter]= var[P[e,g[x1,x2],g[y1,y2],
 g[z1,z2],g[m,n],g[o,p],r[x1,m,y1,z1,n],
 r[x2,o,y2,z2,p]]];
(* (\nabla_\mu R_{\nu\rho\sigma\tau})^2 *)
f[14,counter]= var[P[e,g[x1,x2],g[y1,y2],
 g[z1,z2],g[w1,w2],g[v1,v2],
 r[x1,y1,z1,w1,v1],r[x2,y2,z2,w2,v2]]];

j1=14

Do[f[j,counter+1]=Expand[f[j,counter]],
 {j,1,j1}]
counter=counter+1

(* e and g are covariantly constant. *)
P[x___,d[m_],e,y___]:= P[x,e,d[m],y];
P[x___,d[m_],g[y__],z___]:= P[x,g[y],d[m],z];
(* application of the Leipnitz rule *)
P[x___,d[m_],r[y__],z___]:=
  P[x,r[m,y],z] + P[x,r[y],d[m],z];

(* the end of the derivative *)
Do[f[j,counter+1]=f[j,counter]//.
  P[x___,d[m_]]->0,{j,1,j1}]
counter=counter+1

(* After derivation, the order of the
 factors are irrelevant. *)
Do[f[j,counter+1]=
 f[j,counter]/. P->Times,{j,1,j1}]
counter=counter+1

(* contraction of the metric tensor and
 the curvature tensor *)
Do[f[j,counter+1]=f[j,counter]//.
  {g[m_,n_]*r[x___,n_,y___]->r[x,m,y],
  g[m_,n_]^2->N}, {j,1,j1}]
counter=counter+1

r[x__,m_,m_]:= 0;
r[x__,m_,m_,y_,z_]:= 0;
r[m_,m_,x_,y_]:= 0;

(* contraction of the curvature tensor *)
r[m_,n_,n_,m_]:=  sc;
r[m_,n_,m_,n_]:= -sc;
r[m_,x_,y_,m_]:=  Apply[ri,Sort[{x,y}]];
r[x_,m_,m_,y_]:=  Apply[ri,Sort[{x,y}]];
r[m_,x_,m_,y_]:= -Apply[ri,Sort[{x,y}]];
r[x_,m_,y_,m_]:= -Apply[ri,Sort[{x,y}]];
ri[m_,m_]:= sc;

(* r(5) *)
r[x_,m_,n_,m_,n_]:= -sc[x];
(* Below the same program as Appendix C
 of Ref.1 follows. *)

(* final results for general dimensions,
 e is set to 1. *)
Do[ds[j]=Expand[f[j,counter]/2],{j,1,j1}]
(* 4 dim *)
Do[ds4[j]=ds[j]//. N->4,{j,1,3}]
(* 6 dim *)
Do[ds6[j]=ds[j]//. N->6,{j,4,j1}]

SetDirectory["d:\\trace_anomaly2"]
Save["var_r.txt",ds,ds4,ds6]
```

Appendix B. Trace of Dirac matrices

$$tr1 = 2^{\frac{n}{2}} \tag{B.1}$$

$$tr[\,\gamma^a\gamma^b\,] = 2^{\frac{n}{2}}\eta^{ab} \tag{B.2}$$

$$tr[\,\sigma^{ab}\sigma^{cd}\,] = 2^{\frac{n}{2}-2}\eta^{[ad}\eta^{b]c} \tag{B.3}$$

$$tr[\,\sigma^{ab}\sigma^{cd}\sigma^{ef}\,] = 2^{\frac{n}{2}-3}(\eta^{[ac}\eta^{b][e}\eta^{df]} - \eta^{[ad}\eta^{b][e}\eta^{cf]}) \tag{B.4}$$

$$tr[\,(\hat{R}_{\mu\nu})^2\,] = -2^{\frac{n}{2}-3}(R_{\mu\nu\rho\sigma})^2 \tag{B.5}$$

$$R^{\mu\nu}tr[\,\hat{R}_{\mu\rho}\hat{R}_{\nu}{}^{\rho}\,] = -2^{\frac{n}{2}-3}R^{\mu\nu}R_{\mu\alpha\beta\gamma}R_{\nu}{}^{\alpha\beta\gamma} \tag{B.6}$$

$$R^{\mu\nu\rho\sigma}tr[\,\hat{R}_{\mu\nu}\hat{R}_{\rho\sigma}\,] = -2^{\frac{n}{2}-3}R^{\mu\nu\rho\sigma}R_{\mu\nu\alpha\beta}R^{\alpha\beta}_{\rho\sigma} \tag{B.7}$$

$$tr[\,\hat{R}^{\mu}{}_{\nu}\hat{R}^{\nu}{}_{\rho}\hat{R}^{\rho}{}_{\mu}\,] = 2^{\frac{n}{2}-3}R^{\mu\rho\nu\sigma}R_{\mu\alpha\nu\beta}R_{\rho}{}^{\alpha}{}_{\sigma}{}^{\beta} \tag{B.8}$$

$$tr[\,(D_\mu\hat{R}_{\nu\rho})^2\,] = -2^{\frac{n}{2}-3}(\nabla_\mu R^{\nu\rho\sigma\tau})^2 \tag{B.9}$$

$$tr[\,(D^\mu\hat{R}_{\mu\nu})^2\,] = -2^{\frac{n}{2}-3}(\nabla^\mu R^{\mu\nu\rho\sigma})^2$$
$$= -2^{\frac{n}{2}-2}\{(\nabla_\mu R_{\nu\rho})^2 - \nabla^\mu R_{\nu\rho}\nabla^\nu R_\mu{}^\rho\} \tag{B.10}$$

$$tr[\,\hat{R}^{\mu\nu}D^2\hat{R}_{\nu\rho}\,] = -2^{\frac{n}{2}-3}R^{\mu\nu\rho\sigma}\Box R_{\mu\nu\rho\sigma} \tag{B.11}$$

References and Notes

[1] K. Seo, *Bull. Gifu City Wom. Col.*, No.53, 2004, p.73;

[2] P.B. Gilkey, *J. Differential Geometry*, Vol.10, 1975, p.601. In order to compare our result with Gilkey's one, E in Ref.8 should be substituted by $R/4$. R_{ijij} and R_{ijik} in Ref.8 should be read as the scalar curvature R and the Ricci tensor R_{jk}, respectively.

[3] Miss typings in the computer programs of Ref.1 were found in the course of the present research. The factor 2 in front of $\hat{R}^{\mu}{}_{\nu}\hat{R}_{\mu\rho}\hat{R}^{\nu\rho}$ in Eq.(4.7) was missed in the program to compute the contraction of $I_{\mu_1\cdots\mu_6}$ (Appendix B). The $\xi R(\hat{R}_{\mu\nu})^2$ terms were neglected in the previous programs. Our previous results for d_8 and d_{11} were not correct. Our results after correction of the missing factor 2 and inclusion of the neglected terms are as follows: $d_8 = \frac{1}{144}(-\frac{1}{72} + \frac{\xi}{12}$ for general ξ), and $d_{11} = -\frac{1}{30}$. These results coinside with those of Gilkey in Ref.2. The discrepancy still exist in c_{13} and c_{14}. By the way, there exists a miss typing in Ref.1 The last term of Eq.(4.5) should read as $-R_{\mu\nu}R_{\rho\sigma}R^{\mu\rho\nu\sigma}$.

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