Evaluation of Trace Anomalies in 6 Space-Time Dimensions by Mathematica

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Abstract
Trace anomalies for a scalar and a Dirac field theories in 6 space-time dimensions are evaluated by making use of our result for the third heat kernel coefficient presented in the previous paper. Among 17 possible terms, 7 terms are eliminated by adding local counter terms to the original action. The results for a scalar theory are compared with the results by other people. Keywords: heat kernel, trace anomaly, 6 dimensions, Mathematica

I Introduction
In the previous paper, we present a evaluation method of the third heat kernel coefficient in a Dirac field theory. The trace anomaly of the energy-momentum tensor in 6 space-time dimensions is derivable from the coefficient as follows:

\[
\sqrt{-g} g^{\mu \nu} < T_{\mu \nu}(x) > = -i \lim_{x' \to x} \text{tr} \left[ e^{-\frac{1}{2}Y(x,x')} \delta^{(n)}(x,x') \right], \quad (1.1)
\]

with

\[
Y^{(1/2)} = \mathcal{D}^2 = g^{\mu \nu} D_\mu D_\nu + \xi R, \quad (1.2)
\]

where \( \xi = \frac{1}{4} \). If we retain \( \xi \) as a free parameter, the trace anomaly for the scalar theory,

\[
\sqrt{-g} g^{\mu \nu} < T_{\mu \nu}(x) > = i \lim_{x' \to x} \text{tr} \left[ e^{-\alpha Y^{(0)}} \delta^{(n)}(x,x') \right], \quad (1.3)
\]

with

\[
Y^{(0)} = g^{\mu \nu} \nabla_\mu \nabla_\nu + \xi R = \Box + \xi R, \quad (1.4)
\]

is known by neglecting the derivative terms of \( I \) and setting the parameter \( \xi \) as \( (n-2)/(4(n-1)) \).

The third heat kernel coefficient was evaluated by Gilkey, and Barvinsky et. al. They are giving only the integrated expression instead of the local one, and there exists some ambiguity on the value of the coefficients. In spite of their claim that their results are consistent with Gilkey’s one, their results are rather consistent with our’s.

The trace anomaly in 4 space-time dimensions, there exist 4 possible terms. Among them the term proportional to \( \Box R \) can be eliminated by adding a local counter term. In 6 space-time dimensions, there exist 17 possible terms. First we must know whether these terms can be eliminated by counter terms or not. The contribution to the trace anomaly of the possible local counter terms are evaluated with the help of Mathematica.

Next we select terms to be left and determine their coefficients. Finally we compare our results for a scalar theory with those by other people.

II Elimination of spurious anomalies by local counter terms
Under the local Weyl transformation \( \delta g^{\mu \nu} = 2\alpha g^{\mu \nu} \), the curvature tensor and its covariant derivative transform as follows:

\[
\int d^n x F^\mu \nu \rho \sigma \delta R^\mu \nu \rho \sigma = \int d^n x \alpha \left\{ -2R^\mu \nu \rho \sigma + g_{\nu \rho} \nabla_\sigma \nabla_\nu - g_{\nu \rho} \nabla_\sigma \nabla_\mu - g_{\mu \sigma} \nabla_\rho \nabla_\nu - g_{\mu \sigma} \nabla_\rho \nabla_\nu \right\} F^\mu \nu \rho \sigma, \quad (2.1)
\]

\[
\int d^n x F^\mu \nu \rho \sigma \delta \left( \nabla_\mu R^\nu \rho \sigma \right) = \int d^n x \alpha \left\{ 2 \left( R^\mu \nu \rho \sigma - g_{\nu \rho} \nabla_\sigma \nabla_\nu + g_{\nu \rho} \nabla_\sigma \nabla_\mu \right) F^\mu \nu \rho \sigma \\
+ g_{\mu \sigma} \nabla_\rho \nabla_\nu - g_{\mu \sigma} \nabla_\rho \nabla_\nu \right\} \mu F^\mu \nu \rho \sigma \\
- 4 \nabla_\mu (R^\nu \rho \sigma F^\mu \nu \rho \sigma) - \nabla_\nu (R^\mu \nu \rho \sigma F^\mu \nu \rho \sigma) - \nabla_\sigma (R^\mu \nu \rho \sigma F^\mu \nu \rho \sigma) - \nabla_\tau (R^\mu \nu \rho \sigma F^\mu \nu \rho \sigma) \\
+ g_{\mu \sigma} \nabla_\rho \nabla_\nu (R^\mu \nu \rho \sigma F^\mu \nu \rho \sigma) + g_{\mu \sigma} \nabla_\rho \nabla_\nu (R^\mu \nu \rho \sigma F^\mu \nu \rho \sigma) \right\}, \quad (2.2)
\]

where \( F \)’s are arbitrary tensors. These formulas are coded by Mathematica as shown in Appendix A, and the contributions of local counter terms to the trace anomaly are obtained in the following. For simplicity, we use the following notation,

\[
\Delta \mathcal{L} = \frac{1}{\sqrt{-g}} g^{\mu \nu} \delta \frac{\delta}{\delta g^{\mu \nu}} \int d^n x \sqrt{-g} \mathcal{L}, \quad (2.3)
\]
Table 1: The coefficients \(c\)'s in Eq.(2.14) which describe the contributions of the local counter terms to the trace anomaly.

where \(\mathcal{L}\) is an arbitrary scalar density. In 4 space-time dimensions, there exist 3 possible counter terms and their conformal variations take the following expressions:

\[
\Delta R^2 = 2(1-n)\Box R + (2 - \frac{n}{2})R^2, \tag{2.4}
\]

\[
\Delta (R_{\alpha\beta})^2 = -\frac{n}{2} \Box R + (2 - \frac{n}{2})(R_{\alpha\beta})^2, \tag{2.5}
\]

\[
\Delta (R_{\alpha\beta\gamma\delta})^2 = -2\Box R + (2 - \frac{n}{2})(R_{\alpha\beta\gamma\delta})^2. \tag{2.6}
\]

Therefore the trace anomaly term with the tensor structure \(\Box R\) can be eliminated by the local counter term proportional to \(R^2\). In 6 space-time dimensions, there are 11 possible local counter terms and their conformal variations can eliminate the following 7 tensor structures,

\[
T_1 \equiv \Box^2 R, \tag{2.7}
\]

\[
T_2 \equiv \nabla^\mu(R\nabla_\mu R) = R\Box R + (\nabla_\mu R)^2, \tag{2.8}
\]

\[
T_3 \equiv \nabla_\mu(R^\mu_\nu \nabla_\nu R) = R^\mu_\nu \nabla_\nu \nabla_\mu R + \frac{1}{2}(\nabla^2 R)^2, \tag{2.9}
\]

\[
T_4 \equiv \nabla^\mu(R^\mu_\nu \nabla_\mu R_{\nu\rho}) = R^\nu_\mu \Box R_{\mu} + (\nabla_\mu R_{\nu\rho})^2, \tag{2.10}
\]

\[
T_5 \equiv \nabla^\mu(R^\mu_\nu \nabla_\mu R_{\nu\rho\sigma\tau}) = R^\nu_\mu \Box R_{\nu\rho\sigma\tau} + (\nabla_\mu R_{\nu\rho\sigma\tau})^2, \tag{2.11}
\]

\[
T_6 \equiv \nabla^\mu(R^\mu_\nu \nabla_\mu R_{\nu\rho}) = (\nabla^\nu R^\mu_\rho \nabla_\nu R_{\rho}) + R^\mu_\nu \Box R_{\nu\rho} + (\nabla_\nu R_{\nu\rho})^2, \tag{2.12}
\]

\[
T_7 \equiv \nabla_\mu(R^{\nu\rho\sigma\tau} \nabla_\nu R_{\rho\sigma\tau}) = \nabla_\mu R^{\nu\rho\sigma\tau} \nabla_\nu R_{\rho\sigma\tau} + R^{\nu\rho\sigma\tau} \nabla_\mu \nabla_\nu R_{\rho\sigma\tau}
+ \frac{1}{2}R^{\nu\rho\sigma\tau} \nabla_\nu R_{\rho\sigma\tau} R_{\nu\rho\sigma\tau} + \frac{1}{4}R^{\nu\rho\sigma\tau} \Box R_{\nu\rho\sigma\tau}
+ \frac{1}{2}R^{\nu\rho\sigma\tau} \nabla_\nu R_{\rho\sigma\tau} R^{\nu\rho\sigma\tau} + \frac{1}{4}R^{\nu\rho\sigma\tau} R_{\nu\rho\sigma\tau} R^{\nu\rho\sigma\tau}
+ R_{\nu\rho\sigma\tau} R^{\nu\rho\sigma\tau} R_{\nu\rho\sigma\tau}. \tag{2.13}
\]
The contributions of 11 local counter terms to the trace anomaly are summarized as
\[
\Delta \mathcal{L}_i = \sum_{j=1}^{7} c_{ij} T_j + (3 - \frac{n}{2}) \mathcal{L}_i. \quad (i=1, 2, \cdots 11) \quad (2.14)
\]

The coefficients $c_{ij}$'s are shown in Table 1. Among 11 terms, 7 terms are sufficient to eliminate spurious anomaly terms. For example, $i = 1, 2, 3, 4, 7, 8, 9$.

### III Results and discussions

In the previous section, it is understood that 7 terms can be eliminated by addition of local counter terms. Now we select 10 terms to be left, and we define the 10 coefficients as follows:

\[
\left< T^\mu_\nu(x) \right> = \frac{1}{64\pi^3} \left( a_1 (\nabla_\mu R)^2 + a_2 (\nabla_\nu R_{\rho\sigma\tau})^2 + a_3 R^3 \right. \\
+ a_4 R R_{\mu\nu} + a_5 R R_{\mu\rho\sigma\tau} + a_6 R^2 R_{\mu\nu} R_{\rho\sigma} + a_7 R_{\mu\rho\sigma} R^{\nu\rho\sigma\tau} + a_8 R_{\mu\nu\rho\sigma} R^{\nu\rho\sigma\tau} \\
+ a_9 R_{\mu\nu\rho\sigma} R^{\nu\rho\sigma\tau} \left. \right) . \quad (3.1)
\]

If we denote the coefficients before addition of local counter terms as

\[
\left< T^\mu_\nu(x) \right> \text{bare} = \frac{1}{64\pi^3} \left( a_1 (\nabla_\mu R)^2 + a_2 (\nabla_\nu R_{\rho\sigma\tau})^2 + a_3 R^3 \right. \\
+ a_4 R R_{\mu\nu} + a_5 R R_{\mu\rho\sigma\tau} + a_6 R^2 R_{\mu\nu} R_{\rho\sigma} + a_7 R_{\mu\rho\sigma} R^{\nu\rho\sigma\tau} + a_8 R_{\mu\nu\rho\sigma} R^{\nu\rho\sigma\tau} \\
+ a_9 R_{\mu\nu\rho\sigma} R^{\nu\rho\sigma\tau} \left. \right) . \quad (3.2)
\]

then $c$'s are related with $a$'s and $b$'s as follows:

\[
c_1 = a_1 - b_2 - \frac{1}{2} b_3 + \frac{1}{4} (-b_4 + b_6 + b_7), \quad (3.3)
\]

\[
c_2 = a_2 - b_5 + \frac{1}{4} (-b_4 + b_6 + b_7), \quad (3.4)
\]

\[
c_i = a_i, \quad (i = 3, 4, 5) \quad (3.5)
\]

\[
c_6 = a_6 - b_4 + b_6 + b_7, \quad (3.6)
\]

\[
c_7 = a_7 - b_4 + b_6 + b_7, \quad (3.7)
\]

\[
c_8 = a_8 + \frac{1}{2} (b_4 - b_6), \quad (3.8)
\]

\[
c_9 = a_9 + \frac{1}{4} (b_4 - b_6), \quad (3.9)
\]

\[
c_{10} = a_{10} + b_4 - b_6. \quad (3.10)
\]

<table>
<thead>
<tr>
<th>i</th>
<th>tensor</th>
<th>$c$'s value $(x(-s)^f)$</th>
<th>scalar</th>
<th>Dirac</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$(\nabla_{\mu} R)^2$</td>
<td>$\frac{11}{3500} - \frac{11}{30} + \frac{f}{12}$</td>
<td>-10000</td>
<td>-840</td>
</tr>
<tr>
<td>2</td>
<td>$(\nabla_{\mu} R_{\nu\rho\sigma})^2$</td>
<td>$\frac{1}{3500} - \frac{f}{500}$</td>
<td>-10000</td>
<td>-128</td>
</tr>
<tr>
<td>3</td>
<td>$\left&lt; T^{\mu}_{\nu}(x) \right&gt;$</td>
<td>$\frac{1}{1000} - \frac{f}{1800} - \frac{f^2}{90} + f$</td>
<td>-10000</td>
<td>-270</td>
</tr>
<tr>
<td>4</td>
<td>$R R^{\mu}_{\nu \rho \sigma}$</td>
<td>$\frac{1}{5400} - \frac{f}{1800} - \frac{f^2}{90}$</td>
<td>-10000</td>
<td>-2100</td>
</tr>
<tr>
<td>5</td>
<td>$R^{\mu}_{\nu \rho \sigma}$</td>
<td>$\frac{f}{1500} - \frac{3}{5}$</td>
<td>$\frac{f}{1500} - \frac{3}{5}$</td>
<td>$\frac{f}{1500} - \frac{3}{5}$</td>
</tr>
<tr>
<td>6</td>
<td>$R^{\mu}_{\nu \rho \sigma}$</td>
<td>$\frac{1}{5400} + \frac{f}{1500}$</td>
<td>$\frac{1}{5400} + \frac{f}{1500}$</td>
<td>$\frac{1}{5400} + \frac{f}{1500}$</td>
</tr>
<tr>
<td>7</td>
<td>$R_{\mu \nu \rho \sigma} R^{\nu \rho \sigma}$</td>
<td>$\frac{1}{5400} - \frac{f}{1500}$</td>
<td>$\frac{1}{5400} - \frac{f}{1500}$</td>
<td>$\frac{1}{5400} - \frac{f}{1500}$</td>
</tr>
<tr>
<td>8</td>
<td>$R^{\mu}_{\nu \rho \sigma \tau}$</td>
<td>$\frac{1}{2100} + f$</td>
<td>$\frac{1}{2100} + f$</td>
<td>$\frac{1}{2100} + f$</td>
</tr>
<tr>
<td>9</td>
<td>$R_{\mu \nu \rho \sigma \tau \lambda} R^{\nu \rho \sigma \tau \lambda}$</td>
<td>$\frac{1}{907200} + f$</td>
<td>$\frac{1}{907200} + f$</td>
<td>$\frac{1}{907200} + f$</td>
</tr>
<tr>
<td>10</td>
<td>$R_{\mu \nu \rho \sigma \tau \lambda} R^{\nu \rho \sigma \tau \lambda}$</td>
<td>$\frac{1}{22680} + f$</td>
<td>$\frac{1}{22680} + f$</td>
<td>$\frac{1}{22680} + f$</td>
</tr>
</tbody>
</table>

Table 2: The coefficients $c$'s in Eq.(3.1) which describe the minimal trace anomaly. $f$ is a parameter to discriminate the scalar case and the Dirac case, and taking the value 0 and 1, respectively. The factor $(-8)$ for the Dirac case comes from the trace of the Dirac matrix and the fermion loop.

The formulas for the trace of the Dirac matrix are listed in Appendix B. Our results for the coefficients $c$'s are shown in Table 2.

Bastianelli and Dass\(^6\) have presented a simple method to evaluate the trace anomaly for a scalar theory. Their calculation is based on the paper by Bastianelli, Cuoghi and Nocetti\(^7\), where the consistency condition for the trace anomaly is argued and it is concluded that only 4 coefficients are independent:

\[
< T^\mu_\nu > = \frac{1}{64\pi^3} (a E_0 + c_1 I_1 + c_2 I_2 + c_3 I_3), \quad (3.11)
\]

where $E_0$ is the topological Euler density, and $I$'s are three independent Weyl invariants. In Ref.6, the coefficients have been determined by reducing the prob-
lem to a quantum mechanical one. On the other hand, Ichinose and Ikeda 8) have presented an algorithm to obtain the trace anomaly in higher dimensions, and have carried out the program in a 6 dimensional scalar theory.

In order to compare these results with our’s, we rewrite $E_0$ and 3 Weyl invariants by 17 terms in Eq.(3.2). In fact, 3 terms are absent, $\Box^2 R, (\nabla_\mu R)^2$, and $R^{\mu\nu\rho\sigma} \nabla_\mu R$. We determine 3 coefficients of the contributions of local counter terms to the trace anomaly, $\frac{1}{445} \sum_{i=1}^{17} \alpha_i I_i$, to eliminate these 3 terms. Other 4 coefficients $\alpha_i (i = 3, 4, 5, 7)$ are kept as free parameters. Our results for a scalar theory are shown directly compared with those by other people (however, set symmetric quantum mechanical computation. They, and with those by Ichinose and Ikeda8). Even if we take special values for the parameters $\alpha's$, these results cannot be consistent with each other.

Hatzinikitas and Portugal9) have also given the integrated trace anomaly, by carrying out a supersymmetric quantum mechanical computation. They, however, set $\xi = \frac{2}{5} \lambda$, and their results cannot be directly compared with those by other people ($\xi = \frac{1}{5}$).

Appendix A. Variation of counter terms under the Weyl transformation

SetAttributes[P,Flat];
SetAttributes[g,Orderless];

(* conformal transformation, s is a fixed index.*)
var[P[x__]] := Sum[Apply[P,
Join[{var[x]}/.{}]], Delete[{x},i]];
(* variation of the square root of the metric tensor
var[e] = -NP[e];
(* variation of the metric tensor with upper
 indices *)
var[g[m_,n_,]] := 2*P[g[m,n]];
(* variation of the curvature tensor *)
var[r[m_,n_,o_,p_]] := -2*P[r[m,n,o,p]] +
P[g[m,o],d[p],d[n]] - P[g[m,p],d[o],d[n]] -
P[g[n,o],d[p],d[m]] + P[g[n,p],d[o],d[m]];
(* variation of the derivative of the
curve tensor *)
var[r[m_,n_,o_,p_,q_]] :=
2*P[r[n,o,p,q],d[m]] - P[g[n,p],d[q],
d[o],d[m]] + P[g[n,q],d[p],d[o],d[m]] +
P[g[o,p],d[q],d[m]] - P[g[o,q],d[p],
d[n],d[m]] - 4*P[d[m],r[n,o,p,q]] -
P[d[n],r[m,o,p,q]] - P[d[o],r[m,n,p,q]] -
P[d[p],r[n,o,m,q]] - P[d[q],r[n,o,p,m]] +
P[g[m,n],d[s],r[s,o,p,q]] + P[g[m,o],d[s],
r[s,p,q]] + P[g[m,p],d[s],r[n,o,s,q]] +
P[g[m,q],d[s],r[n,o,p,s]]];

<table>
<thead>
<tr>
<th>tensor</th>
<th>symbol</th>
<th>our results</th>
<th>B.D.</th>
<th>I.I.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R \Box R$</td>
<td>(sc*ddsc)</td>
<td>$\frac{1}{25200} - \frac{\alpha_2}{2}$</td>
<td>$\frac{1}{4200}$</td>
<td>$-\frac{11}{18900}$</td>
</tr>
<tr>
<td>$R^{\mu\nu} \Box R_{\mu\nu}$</td>
<td>(rdd1)</td>
<td>$\frac{1}{25200} + \frac{\alpha_4}{4}$</td>
<td>$-\frac{1}{4200}$</td>
<td>$-\frac{11}{18900}$</td>
</tr>
<tr>
<td>$R^{\mu\nu\rho\sigma} \Box R_{\mu\nu\rho\sigma}$</td>
<td>(ccdc)</td>
<td>$\frac{1}{25200} + \alpha_4 + \frac{\alpha_7}{4}$</td>
<td>$-\frac{1}{840}$</td>
<td>$0$</td>
</tr>
<tr>
<td>$(\nabla_\rho R_{\mu\nu})^2$</td>
<td>(dssq2)</td>
<td>$\frac{1}{25200} + \alpha_4 + \alpha_7$</td>
<td>$-\frac{1}{840}$</td>
<td>$0$</td>
</tr>
<tr>
<td>$(\nabla_\rho R_{\mu\nu\rho\sigma})^2$</td>
<td>(dssq3)</td>
<td>$\frac{1}{25200} + \alpha_4 + \alpha_7$</td>
<td>$-\frac{1}{840}$</td>
<td>$0$</td>
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<tr>
<td>$R^3$</td>
<td>(sc^3)</td>
<td>$\frac{1}{10200} + \frac{\alpha_5}{2}$</td>
<td>$\frac{1}{16200}$</td>
<td>$47$</td>
</tr>
<tr>
<td>$R^2 R_{\mu\nu}$</td>
<td>(sc^rsq)</td>
<td>$\frac{1}{5400}$</td>
<td>$\frac{1}{5400}$</td>
<td>$17$</td>
</tr>
<tr>
<td>$R^2 R_{\mu\nu\rho\sigma}$</td>
<td>(sc^csq)</td>
<td>$\frac{1}{5400}$</td>
<td>$\frac{1}{5400}$</td>
<td>$11$</td>
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<tr>
<td>$R_{\mu\nu} R_{\rho\sigma} R^{\rho\sigma\tau\lambda}$</td>
<td>(rcu)</td>
<td>$\frac{1}{111340} + 2\alpha_3$</td>
<td>$\frac{1}{18900}$</td>
<td>$5670$</td>
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<tr>
<td>$R_{\mu\nu\rho\sigma} R_{\rho\sigma\tau\lambda}$</td>
<td>(crsq)</td>
<td>$\frac{1}{111340} + \frac{\alpha_6}{2}$</td>
<td>$\frac{1}{1770}$</td>
<td>$\frac{1}{1260}$</td>
</tr>
<tr>
<td>$R_{\mu\nu\rho\sigma\tau\lambda} R^{\rho\sigma\tau\lambda}$</td>
<td>(cscsq)</td>
<td>$\frac{1}{111340} + \frac{\alpha_6}{4}$</td>
<td>$\frac{1}{7500}$</td>
<td>$\frac{1}{126}$</td>
</tr>
</tbody>
</table>

Table 3: Comparison of the coefficients of the trace anomaly for a scalar theory. B.D. and I.I. are the results of Ref.6 and Ref.8, respectively.

(* distribution rule and multiplication of
counter terms *)
1. $P[x_\ldots,y_1\ldots2\ldots,\ldots,z_\ldots\ldots] = P[x,y_1,z] + P[x,y_2,z]$;
2. $P[x_\ldots,c_\ldots,P[y_\ldots],z_\ldots\ldots] = c_\ldots P[x,y_1,z]$;

counter=1
(* R^2 *)
f[1,c] := var[P[e,g[m,n],g[o,p],r[m,o,p,n],g[a,b],g[c,d],r[a,c,d,b]]];
f[2,c] := var[P[e,g[m,n],g[o,p],g[o1,p1],g[o2,p2],r[m1,o1,p1,n1],r[m2,o2,p2,n2]]];
(* R_\ldots \mu \ldots * mula *)
f[3,c] := var[P[e,g[m1,n2],g[n1,n2],g[o1,p1],g[o2,p2],r[m1,o1,p1,n1],r[m2,o2,p2,n2]]];
(* R_\ldots \mu \ldots \rho \ldots \sigma \ldots \ldots *)
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\begin{align*}
&f[4, \text{counter}] = \text{var}[P[e, g[m_1, n_1, g[o_1, p_1], r[m_1, o_1, p_1, g[m_2, n_2, g[o_2, p_2], r[m_2, o_2, p_2, n_2], r[m_3, o_3, p_3, n_3]]]]; \\
&\text{(\textit{\textbf{R^3}})}
\end{align*}

\begin{align*}
&f[5, \text{counter}] = \text{var}[P[e, g[m_1, m_2, g[n_1, n_2, g[o_1, p_1, g[p_1, p_2], g[m_3, n_3, g[o_3, p_3, r[m_1, o_1, p_1, n_1, r[m_2, o_2, p_2, n_2], r[m_3, o_3, p_3, n_3]]]]]; \\
&\text{(\textit{\textbf{R(R_{\mu\mu})^2}})}
\end{align*}

\begin{align*}
&f[6, \text{counter}] = \text{var}[P[e, g[m_1, m_2, g[n_1, n_2, g[o_1, o_2, g[p_1, p_2], g[m_3, n_3, g[o_3, p_3, r[m_1, o_1, p_1, n_1, r[m_2, o_2, p_2, n_2], r[m_3, o_3, p_3, n_3]]]]]; \\
&\text{(\textit{\textbf{R(R_{\mu\nu\rho\sigma})^2}})}
\end{align*}

\begin{align*}
&f[7, \text{counter}] = \text{var}[P[e, g[m_1, m_2, g[n_1, m_3, g[o_1, p_1, g[p_2, p_3], g[o_3, p_3, r[m_1, o_1, p_1, n_1, r[m_2, o_2, p_2, n_2], r[m_3, o_3, p_3, n_3]]]]]; \\
&\text{(\textit{\textbf{R_{\mu\nu}R_{\nu\rho}R_{\rho\mu}})}
\end{align*}

\begin{align*}
&f[8, \text{counter}] = \text{var}[P[e, g[m_1, m_3, g[n_1, m_2, g[n_2, n_3, g[p_1, p_2], g[p_2, p_3], r[m_1, o_1, p_1, n_1, r[m_2, o_2, p_2, n_2], r[m_3, o_3, p_3, n_3]]]]]; \\
&\text{(\textit{\textbf{R_{\mu\nu}R_{\rho\sigma}R_{\mu\rho\nu\sigma}})}
\end{align*}

\begin{align*}
&f[9, \text{counter}] = \text{var}[P[e, g[m_1, m_2, g[n_1, m_3, g[n_2, n_3, g[p_2, p_3], g[p_2, p_3], r[m_1, o_1, p_1, n_1, r[m_2, o_2, p_2, n_2], r[m_3, o_3, p_3, n_3]]]]]; \\
&\text{(\textit{\textbf{R_{\mu\nu}R_{\mu\alpha\beta\gamma}R_{\nu\alpha\beta\gamma}})}
\end{align*}

\begin{align*}
&f[10, \text{counter}] = \text{var}[P[e, g[m_1, m_2, g[n_1, o_3, g[n_2, n_3, g[p_2, p_3], g[p_2, p_3], r[m_1, o_1, p_1, n_1, r[m_2, o_2, p_2, n_2], r[m_3, o_3, p_3, n_3]]]]]; \\
&\text{(\textit{\textbf{R_{\mu\rho\nu\sigma}R_{\mu\alpha\nu\beta}R_{\rho\alpha\sigma\beta}})}
\end{align*}

\begin{align*}
&f[11, \text{counter}] = \text{var}[P[e, g[x, y], g[m, n], g[o, p], r[x, m, o, p, n], g[a, b], g[c, d], r[y, a, c, d, b]]]; \\
&\text{(\textit{\textbf{\nabla_{\mu}R}})}
\end{align*}

\begin{align*}
&f[12, \text{counter}] = \text{var}[P[e, g[x, y], g[m, n], g[o, p], r[x, m, o, p, n], g[a, b], g[c, d], r[y, a, c, d, b]]]; \\
&\text{(\textit{\textbf{\nabla_{\rho\sigma}R}})}
\end{align*}

\begin{align*}
&f[13, \text{counter}] = \text{var}[P[e, g[x_1, x_2, g[y_1, y_2], g[z_1, z_2], g[m, n], g[o, p], r[x_1, y_1, z_1, n], r[x_2, y_2, z_2, p]]]; \\
&\text{(\textit{\textbf{\nabla_{\rho\sigma\tau\rho\tau}})}
\end{align*}

\begin{align*}
&f[14, \text{counter}] = \text{var}[P[e, g[x_1, x_2], g[y_1, y_2], g[z_1, z_2], g[w_1, w_2], g[v_1, v_2], r[x_1, y_1, z_1, w_1, v_1], r[x_2, y_2, z_2, w_2, v_2]]]; \\
&\text{(\textit{\textbf{\nabla_{\mu\nu\rho\sigma\tau\tau}})}
\end{align*}
Appendix B. Trace of Dirac matrices

\begin{equation}
tr 1 = 2^\frac{1}{2}
\end{equation}

\begin{equation}
tr[\gamma^a\gamma^b] = 2^\frac{1}{2}\eta^{ab}
\end{equation}

\begin{equation}
tr[\sigma^{ab}\sigma^{cd}] = 2^{\frac{5}{2}} - 2\eta^{[ad}\eta^{bc]}
\end{equation}

\begin{equation}
tr[\sigma^{ab}\sigma^{cd}\sigma^{ef}] = 2^{\frac{7}{2}} - 3\eta^{[ae}\eta^{bf]} - \eta^{[ad}\eta^{be]}
\end{equation}

\begin{equation}
tr[(\hat{R}_{\mu\nu})^2] = -2^{\frac{3}{2}} - 3(R_{\mu\rho\sigma\rho})^2
\end{equation}

\begin{equation}
R^{\mu\nu}tr[\hat{R}_{\mu\nu}\hat{R}_{\rho\sigma}] = -2^{\frac{3}{2}} - 3R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}
\end{equation}

\begin{equation}
tr[\hat{R}_{\nu\rho}d\hat{R}_{\mu\rho}] = 2^{\frac{3}{2}} - 3R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}
\end{equation}

\begin{equation}
tr[(D_{\mu}\hat{R}_{\nu\rho})^2] = -2^{\frac{3}{2}} - 3(\nabla_{\mu}R^{\nu\rho\sigma\tau})^2
\end{equation}

\begin{equation}
tr[(D_{\mu}\hat{R}_{\nu})^2] = -2^{\frac{3}{2}} - 3(\nabla_{\mu}R^{\nu\rho\sigma\tau})^2
\end{equation}

\begin{equation}
= -2^{\frac{3}{2}} - 2(\nabla_{\mu}R_{\nu\rho\sigma}\nabla_{\mu}R_{\nu\rho\sigma})
\end{equation}

\begin{equation}
tr[\hat{R}_{\mu\nu}D^2\hat{R}_{\rho\sigma}] = -2^{\frac{3}{2}} - 3R^{\mu\nu\rho\sigma}D^2R_{\mu\nu\rho\sigma}
\end{equation}

References and Notes


[2] P.B. Gilkey, J. Differential Geometry, Vol.10, 1975, p.601. In order to compare our result with Gilkey’s one, E in Ref.8 should be substituted by R/4. R_{ijj} and R_{ijk} in Ref.8 should be read as the scalar curvature R and the Ricci tensor R_{jk}

[3] Miss typos in the computer programs of Ref.1 were found in the course of the present research. The factor 2 in front of $\hat{R}_{\nu\rho}d\hat{R}_{\mu\rho}$ in Eq.(4.7) was missed in the program to compute the contraction of $I_{\nu\mu\nu\rho}$(Appendix B). The $\xi R(R_{\mu\nu})^2$ terms were neglected in the previous programs. Our previous results for $d_8$ and $d_11$ were not correct. Our results after correction of the missing factor 2 and inclusion of the neglected terms are as follows: $d_8 = \frac{1}{11}(-\frac{1}{12} + \frac{1}{14}$ for general $\xi)$, and $d_{11} = -\frac{1}{10}$. These results coincide with those of Gilkey in Ref.2. The discrepancy still exist in $c_{13}$ and $c_{14}$. By the way, there exists a miss typing in Ref.1 The last term of Eq.(4.5) should read as $-R_{\mu\nu}R_{\rho\sigma}R^{\mu\nu\rho\sigma}$.


